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ANALYTICAL TREATMENT OF RESISTIVE WAKE POTENTIALS IN ROUND PIPE

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Abstract

The modified analytical form for the longitudinal and transverse resistive wake potentials of point-like charge moving parallel to the axis of round pipe with frequency independent walls conductivity is obtained. The short range wake potentials are presented by uniformly converged series.

For the frequency dependent conductivity, the resonator term of the longitudinal monopole wake potential is presented in analytical form. The diffusion term of the potential is modified to simple integral form.

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1. INTRODUCTION

The longitudinal and transverse impedances of infinitely resistive round pipe with frequency independent conductivity have been obtained by Chao [1]. The further treatment of the impedances both for frequency independent (DC-direct current) and frequency dependent (AC-alternate current) conductivities has been performed in Ref. [2].

In particular, the analytical extension of longitudinal impedance (monopole term) to the complex plane has been derived as [2]:

$$Z(k) = \frac{2s_0}{cb^2}\left[\frac{l}{k} - \frac{ik}{2}\right]^{-1}, \tag{1}$$

where b is the pipe radius, $s_0 = (cb^2/2p s)^{1/3}$ is the characteristic distance, c is the velocity of light, s is the conductivity of the walls and $k = ks_0$ is the complex dimensionless wavenumber.

For the DC conductivity, s is the static quantity and the parameter l is given by

$$l = \sqrt{k}(1+i). \tag{2}$$

For the AC case, the conductivity is given by $s = s_0/(1-iwt)$ with s_0 the static conductivity, w the frequency and t the relaxation time of the metal. The parameter l is then given by [2]:

$$l = k^{1/2}\left(1+k^2\Gamma^2\right)^{-1/4}\left[i\sqrt{1+t_l} + \sqrt{1-t_l}\right] \tag{3}$$

with

$$t_l = \frac{k\Gamma}{\sqrt{1+k^2\Gamma^2}}, \qquad \Gamma = \frac{ct}{s_0}. \tag{4}$$

The complex plane has a brunch cut on the negative imaginary axis. The function confined to the Riemann sheet $-p/2 < q < 3p/2$ and parameter l has different signs in opposite sides of the brunch cut.

The wake potential is given by the inverse Fourier transformation of the longitudinal impedance:

$$W_z(s) = \frac{c}{2ps_0}\int_{-\infty}^{\infty} Z(k)e^{-ik\, s/s_0}dk\ . \tag{5}$$

For the DC case, the integration of (5) in the complex plane results on the sum of resonator (given by elementary functions) and diffusion (given in the integral form) terms [2]. For the AC conductivity, the wake function is given by the numerical integration of (5) [2,3]. The numerical integration of the resonator term in addition implies the numerical determination of impedance function poles.

In this paper, the further treatment of the wake potentials for both DC and AC conductivities has been performed. For the DC conductivity, the analytical form of the diffusion term has been obtained for both longitudinal and transverse wake potentials. For the AC conductivity case, the analytical form for the resonator term and the simple calculating integral for diffusion term of the longitudinal monopole wake potential is derived.

2. DC CONDUCTIVITY

2.1 Point-Charge

The longitudinal and transversal wake potentials for the point driving charge may be presented in the form of expansion over the longitudinal $(w_{z.n})$ and transversal $(\vec{w}_{r.n})$ multipole moments [1,2]:

$$w_z(s,\vec{r}) = \sum_{n=0}^{\infty} w_{z,n}(s,\vec{r}),\ \ \vec{w}_r(s,\vec{r}) = \sum_{n=1}^{\infty} \vec{w}_{r,n}(s,\vec{r}), \tag{6}$$

with

$$w_{z,n}(s,\vec{r}) = \frac{4}{b^2}\left(\frac{rr_1}{b^2}\right)^n d_n g_n f_z(u_n)\cos n(f - f_1),$$

$$\vec{w}_{r,n}(s,\vec{r}) = \frac{4s_0 n}{b^2 r}\left(\frac{rr_1}{b^2}\right)^n g_n^{1/3} f_r(u_n)\left(\cos n(f - f_1)\vec{e}_r - \sin n(f - f_1)\vec{e}_f\right) \tag{7}$$

where $d_0 = g_0 = 1$, $d_{n>0} = 2$, $g_{n>0} = (n+1)/2$, f_1, r_1 and f, r are the cylindrical coordinates of driving and test particles respectively, $\vec{e}_f, \vec{e}_r$ are the unit vectors. The argument u_n is equal to $g_n^{2/3} s/s_0$.

The representation (7) allows express the dependence of the potentials from longitudinal coordinate s for any multipole moments via the longitudinal $f_z(u)$ and transverse $f_r(u)$ wake functions with modified argument $u = u_n$ for n-th multipole. The wake functions f_z, f_r for any multipole moments are given by the sum of resonator and diffusion terms [2]

$$\begin{aligned} f_z(u) &= -\frac{4}{3}e^{-u}\cos\left(\sqrt{3}u\right) + \frac{4\sqrt{2}}{p} I_z(u) \\ f_r(u) &= \frac{2}{3}e^{-u}\left(\sqrt{3}\sin(u) - \cos\left(\sqrt{3}u\right)\right) - \frac{8\sqrt{2}}{p} I_r(u), \end{aligned} \tag{8}$$

with the diffusion integrals I_z, I_r given as

$$I_z(u) = \int_0^{\infty} \frac{x^2 e^{-x^2 u}}{x^6 + 8}dx, \quad I_r(u) = \int_0^{\infty} \frac{e^{-x^2 u}}{x^6 + 8}dx. \tag{9}$$

To transform integrals (9) to analytical forms, the integrands can be rewritten as:

$$\frac{x^2}{x^6 + 8} = \sum_{j=1}^{3} \frac{A_j^1}{x^2 + (B_j)^2}, \quad \frac{1}{x^6 + 8} = \sum_{j=1}^{3} \frac{A_j^2}{x^2 + (B_j)^2},$$

$$A_1^1 = -2A_1^2 = -1/6,\ A_{2,3}^1 = -2A_{2,3}^{2\ *} = (1 \mp i\sqrt{3})/12, \tag{10}$$

$$B_1 = \sqrt{2}, \quad B_{2,3} = (1 \mp i\sqrt{3})/\sqrt{2}.$$

After integration, the wake functions $f_z,\ f_r$ can be presented as:

$$\begin{aligned} f_z(s) &= \left\{-4e^{-s}\cos(\sqrt{3}s) - x(i\sqrt{2s}) + x(e^{i\pi/6}\sqrt{2s}) + x(-e^{-i\pi/6}\sqrt{2s})\right\}/3,) \\ f_r(s) &= \left\{2e^{-s}\left(\sqrt{3}\sin(\sqrt{3}s) - \cos(\sqrt{3}s)\right) + x(i\sqrt{2s}) + e^{-i\pi/6}x(e^{i\pi/6}\sqrt{2s}) + e^{i\pi/6}x(-e^{-i\pi/6}\sqrt{2s})\right\}/3, \end{aligned} \tag{11}$$

where $x(s) = \exp(-s^2)\, erfc(-is)$ is the complex error function [4].

The analytical form of the wake functions $f_z(s)$ and $f_r(s)$ can be modified by expanding the expressions (11) into the series [5]:

$$\begin{aligned} f_z(s) &= \sum_{k=0}^{\infty} c_k s^{3k} + \sum_{k=1}^{\infty} d_k s^{3k-3/2}, \\ f_r(s) &= s\sum_{k=0}^{\infty} \tilde{c}_k s^{3k} + s\sum_{k=1}^{\infty} \tilde{d}_k s^{3k-3/2}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} c_k &= -2^{3k}/(3k!),\ \tilde{c}_k = 2^{3k+1}/(3k+1)!, \\ d_k &= \sqrt{2/\pi}\ 2^{6k-3}/(6k-3)!!, \\ \tilde{d}_k &= -\sqrt{2/\pi}\ 2^{6k-1}/(6k-1)!!. \end{aligned} \tag{13}$$

The expansions (12) are uniformly converged at the interval $0 \le s < \infty$ and are the Laibnitz series, i.e. the consecutive terms of series (12) have alternative signs ($d_k < 0, c_k > 0, d_{k+1} < 0$) and decrease by the module with the term number. Fig.1 presents the longitudinal $f_z(u)$ and transverse $f_r(u)$ wake functions given by (11) that are exactly coincide with the integral representation (8) (solid curves). The dashed curves show the wake functions approach by series expansion (5, 10 and 15 terms in series). Note, that the argument of the wake functions for n-th multipole moment is given by $u = u_n = \gamma_n^{3/3}\, s/s_0$.

2.2. Gaussian Bunch

For the arbitrary longitudinal distribution of the driving bunch, the wake potentials are given by the convolution of the point-charge wake potentials and bunch distribution. In particular, for the Gaussian bunch wake function $F_{z,r}(z)$ we get:

$$F_{z,r}(z) = (2p)^{-1/2}\int_0^{\infty} f_{z,r}(\tilde{s}\,z)\,e^{-\frac{(\tilde{z}-\tilde{s})^2}{2}}\,d\tilde{s}\,. \tag{14}$$

where $\tilde{s}_0 = s_0/g_n^{3/2}$, $\tilde{z} = z/s_z$, $\tilde{s} = s/s_z$, $z = s_z/\tilde{s}_0$ and s_z is the bunch rms length. The expansions of the wake functions for the Gaussian bunch may be then obtained by putting the point-

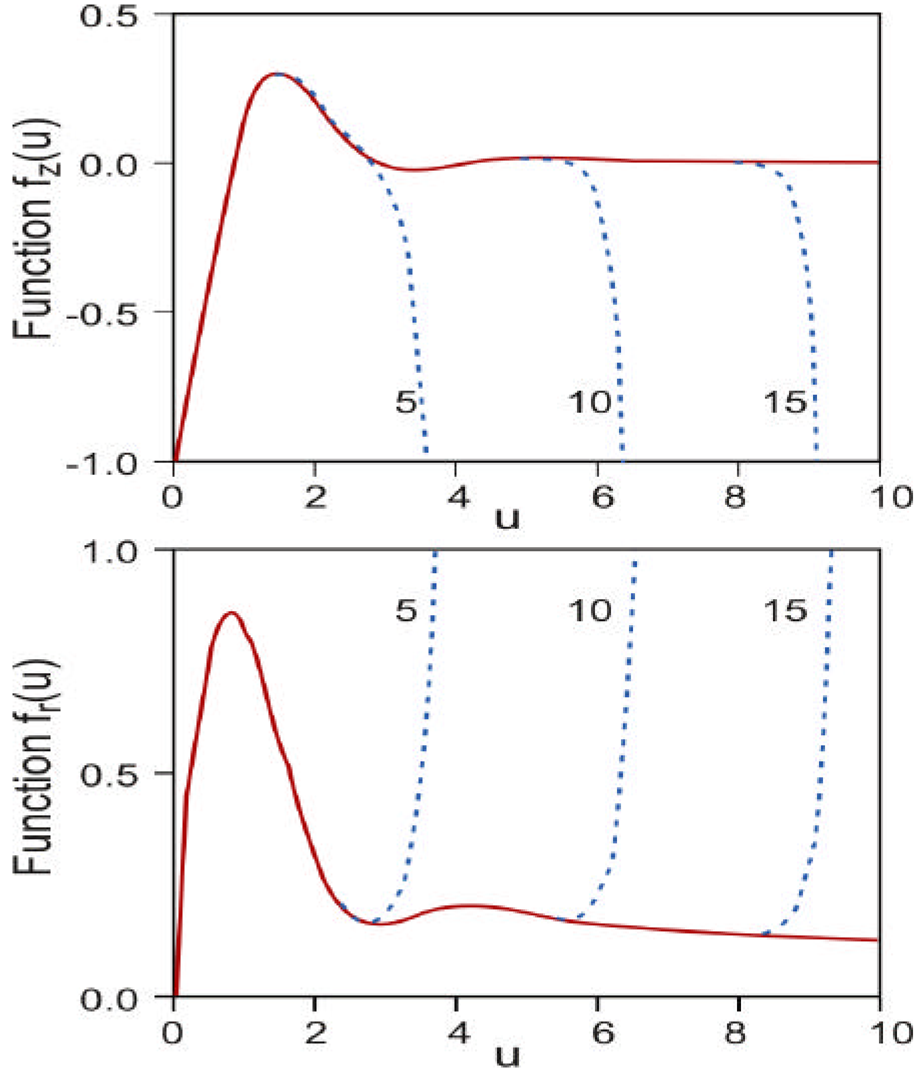

Figure 1. *Wake functions* $f_z(u)$ *(top) and* $f_r(u)$ *(bottom). Solid lines are the results of numerical integration. Dashed lines are the serial approach with 5, 10 and 15 terms of expansion.*

charge wake function series (12) into expression (14). The longitudinal wake function of Gaussian bunch is then expressed as

$$F(z) = \frac{e^{-z^2/4s_z^2}}{\sqrt{2p}}\sum_{k=0}^{\infty}\left\{c_k z^{3k}\Gamma(3k+1)U\left(3k+\frac{1}{2}, -z/s_z\right) + d_k z^{3k-3/2}\Gamma(3k-1/2)U(3k-1,-z/s_z)\right\} \tag{15}$$

where $U(a,x)$ is a function of parabolic cylinder [4]. The several first terms of this expansion give a good coincidence with the directly integrated expression (14) for the case of comparatively small z $(z<1)$. Fig.2 presents the approximation of the Gaussian bunch wake function (solid line) by the serial expansion (5, 10, 15 terms, dashed lines) for $z=0.5$. Actually, the approximation curves detached from the exact solution at different distance behind the bunch depending on the number of terms in series.

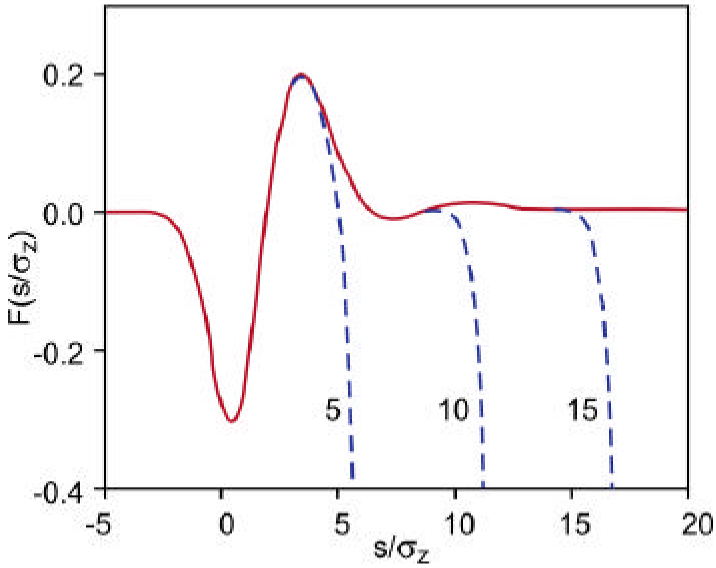

Figure 2. The longitudinal resistive wake function for Gaussian bunch with $s_z/s_0=0.5$. The solid curve is an exact solution; the dashed curves are the wake functions computed by expansion (15) for various numbers of first terms in the sum.

For $V=\tilde{s}_0/s>1$ the solution may be obtained by substituting (11) into (14). After successive partial integration, the result is given in a series expansion by positive degree of $\tilde{s}_0/s_z$:

$$F(\tilde{z})=-\sum_{k=1}^{\infty}\frac{z^{-3k}}{2^{3k}}\left\{z^{3/2}\frac{d^{3k-2}}{d\tilde{z}^{3k-2}}Q_{\pm}(\tilde{z})-\frac{1}{\sqrt{2p}}\frac{d^{3k-1}}{d\tilde{z}^{3k-1}}e^{-\frac{\tilde{z}^2}{2}}\right\},$$

$$Q_{\pm}(z)=\sqrt{\pm z}e^{-z^2/4}\left\{I_{-1/4}\left(z^2/4\right)\pm I_{1/4}\left(z^2/4\right)\right\} \qquad (16)$$

with positive sign for $\tilde{z}>0$ and negative one for $\tilde{z}<0$; $I_{\pm 1/4}(z)$ are the modified Bessel functions [4]. In power of parameter z the wake function $F(z)$ can be presented as:

$$F(\tilde{z})=A_1z^{-3/2}+A_2z^{-3}+A_3z^{-9/2}+A_4z^{-6}+\ldots \qquad (17)$$

The first term of this expansion coincides with the well-known Piwinski formula [6] for the longitudinal wake function in low frequency approximation, i.e.

$$A_1 = -\frac{(\pm \tilde{z})^{3/2}}{2^4} e^{-u} \{ - I_{1/4}(u) + I_{-3/4}(u) \mp I_{-1/4}(u) \pm I_{3/4}(u) \}, \quad u = \tilde{z}^2/4. \tag{18}$$

The next three terms in expansion are given below.

$$A_2 = \frac{1}{2^3\sqrt{2\pi}} (\tilde{z}^2 - 1) e^{-\tilde{z}^2/2},$$

$$A_3 = -\frac{\sqrt{\pm \tilde{z}}}{2^8} e^{-\tilde{z}^2/4} \{ \tilde{f}(\tilde{z}) (\mp I_{-3/4}(u) - I_{3/4}(u)) + + \tilde{g}(\tilde{z}) (I_{-1/4}(u) \pm I_{1/4}(u)) \}, \tag{19}$$

$$A_4 = -\frac{1}{2^6\sqrt{2\pi}} \tilde{z} (15 - 10\tilde{z}^2 + \tilde{z}^4) e^{-\tilde{z}^2/2},$$

where

$$\tilde{f}(z) = 2z^2(z^2 - 4), \quad \tilde{g}(z) = 5 + 2z^2(z^2 - 5). \tag{20}$$

Fig.3 presents the longitudinal wake functions for the Gaussian bunch for $z = 2.5$ obtained by exact integration and series expansion given by (16), (17).

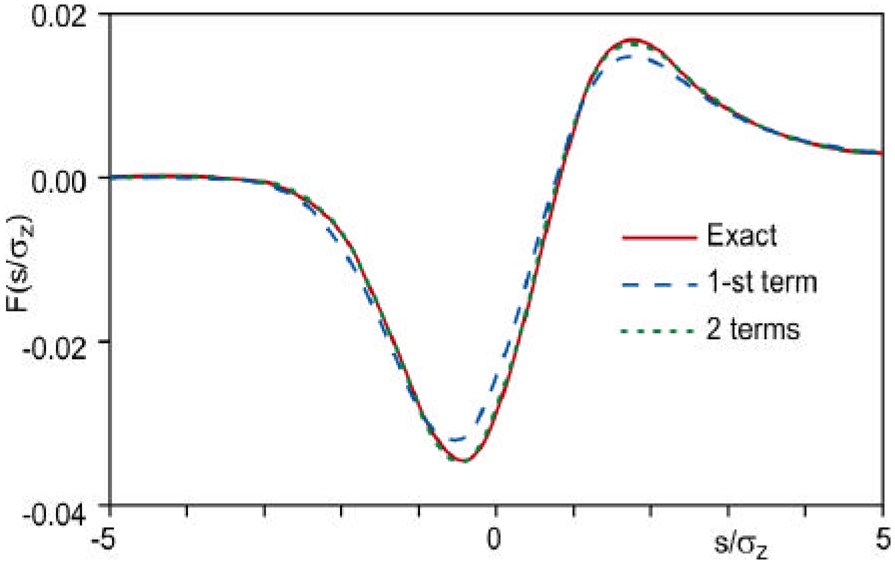

Figure 3. The longitudinal wake function for $z = 2.5$. Exact solution (solid line) and its approximation by first (dashed line) and two (dotted line) terms of expansion (17) are shown.

A good convergence is observed already with two terms of series expansion. Thus for $z \geq 2.5$, the two first terms of (17), i.e. the Piwinski formula plus the second term of expansion, with high accuracy describe the Gaussian bunch wake function:

$$F_s(\tilde{z}) = \frac{1}{\sqrt{2p}}\left\{\frac{(\pm\tilde{z})^{3/2}}{16z^{3/2}}e^{-u}\left\{I_{1/4}(u) - I_{-3/4}(u) \pm I_{-1/4}(u) \mp I_{3/4}(u)\right\} + \frac{1}{8z^3}\frac{1}{\sqrt{2p}}\left(\tilde{z}^2 - 1\right)e^{-2u}\right\} \quad (21)$$

For more precise calculation of wake potential the next terms may be used. On Fig.4 in enlarged scale the consecutive convergence of the expansion (17) to the exact solution by using 2, 3 and 4 terms of expansion are given.

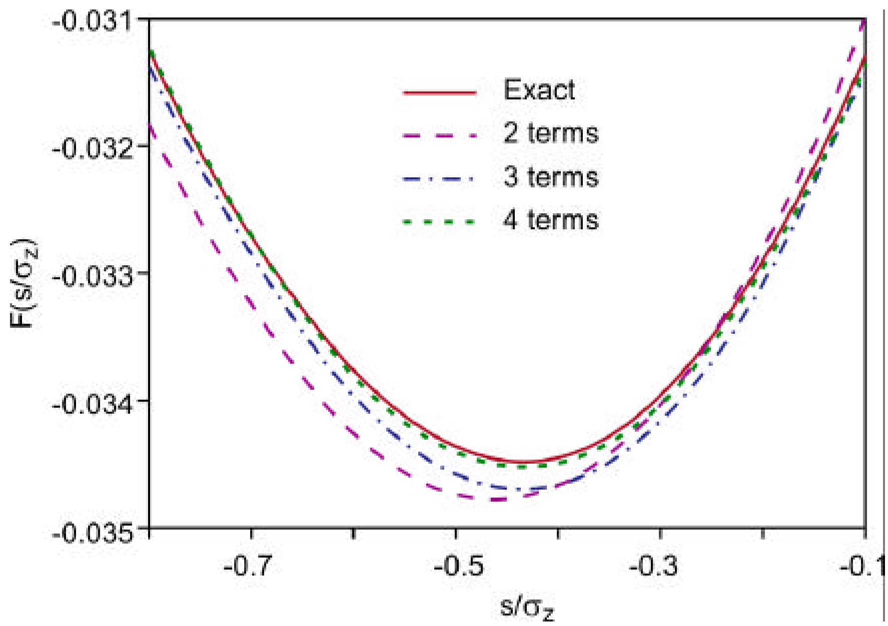

Figure 4. The convergence of approximation (17) to the exact solution for $z = 2.5$.

3. AC CONDUCTIVITY

The longitudinal impedance (1) for the AC conductivity after change of variables $\tilde{l} = -il$, $k = i\tilde{l}^2/(2 - \tilde{l}^2\Gamma)$ and $\Gamma = ct/s_0$, may be rewritten as:

$$Z = -\frac{2s_0}{cb^2}\frac{\tilde{l}(2 - \tilde{l}^2\Gamma)}{(2 - \tilde{l}^2\Gamma)^2 + \tilde{l}^3/2} \quad (22)$$

The denominator of (22) has four roots given by:

$$\tilde{l}_{1,2} = -1/8\Gamma^2 - g/2\Gamma^2 \mp e_-/2\Gamma^2,$$
$$\tilde{l}_{3,4} = -1/8\Gamma^2 + g/2\Gamma^2 \mp e_+/2\Gamma^2, \quad (23)$$

where

$$g = \sqrt{a_1 + a_2}, \quad e_\mp = (2a_1 - a_2 \mp a_3/4g)^{1/2} \quad (24)$$

with

$$a_1 = \frac{1}{16} + \frac{8\Gamma^3}{3}\ ,\ \ a_2 = \frac{\Gamma^2}{3}\left(\frac{64\Gamma^2}{d} + d\right),\ \ a_3 = -\frac{1}{8} - 8\Gamma^3, \tag{25}$$

and

$$d = 2^{-1/3}\left(27 + 1024\Gamma^3 + 3\sqrt{3}\sqrt{27 + 2048\Gamma^3}\right)^{1/3}. \tag{26}$$

The first two roots $\tilde{l}_{1,2}$ are real while $\tilde{l}_{3,4}$ are complex (the term e_+ is imaginary for arbitrary $\Gamma > 0$). In DC limit $(\Gamma = 0)$: $\tilde{l}_1 = -\infty$, $\tilde{l}_2 = -2$ and $\tilde{l}_{3,4} = 1 \mp i\sqrt{3}$. The last two roots give a negative imaginary part for the wavenumber k :

$$k_{3,4} = i\tilde{l}_{3,4}^2\left(2 - \tilde{l}_{3,4}^2\Gamma\right)^{-1} = \mp\frac{a}{M} - i\frac{b}{M} \tag{27}$$

where $a = 4uw$, $b = f_+^2\Gamma - 2f_-$ with $f_\pm = u^2 \pm w^2$ and $M = 4 - 4\Gamma f_- + \Gamma^2 f_+^2$.

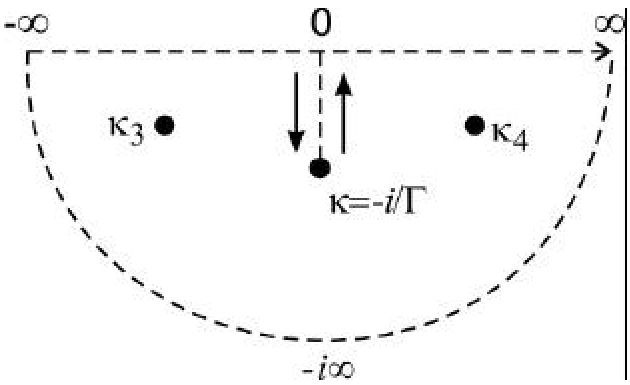

Figure 5. The contour of integration.

The integration contour for inverse Fourier transformation to obtain the longitudinal wake potential is presented in Fig. 5. Similar to the DC conductivity, the wake function is presented by the difference of resonator and diffusion terms. The resonator term is given by the integration over the complete contour (sum of the residues), while the diffusion term is given by the integral along the cut of. The resonant term of the longitudinal wake potential, given by the sum of two residues at $k = k_3$ and $k = k_4$, is then presented as

$$W_z^R(s) = -\frac{8}{b^2}\sum_{j=3,4}\frac{(2 - \tilde{l}_j^2\Gamma)^{-1}\tilde{l}_j e^{-ik_j s/s_0}}{4\Gamma^2\tilde{l}_j^2 + 3\tilde{l}_j/2 - 8\Gamma}. \tag{28}$$

In terms of real variables the resonator term (28) may be rewritten as:

$$W_z^R(s) = -\frac{8}{b^2 MQ}e^{-\frac{b}{M}\frac{s}{s_0}}\left\{F_c\cos\frac{a}{M}\frac{s}{s_0} + F_s\sin\frac{a}{M}\frac{s}{s_0}\right\} \tag{29}$$

where

$$Q = 16\Gamma^2 M - 12\Gamma u(2-\Gamma f_+) + \frac{9}{4} f_+,$$
$$F_c = 32\Gamma u(\Gamma f_+ - 1) - 8\Gamma^3 f_+ u\left(f_- - 2w^2\right) + 3 f_+ (2 - \Gamma f_-), \qquad (30)$$
$$F_s = 32\Gamma w(\Gamma f_+ + 1) - 8\Gamma^3 f_+ w\left(f_- + 2u^2\right) - 6\Gamma f_+ uw.$$

Thus in AC conductivity case, the resonator term (29) of the longitudinal wake potential is the superposition of two waves shifted by the quarter wavelength.

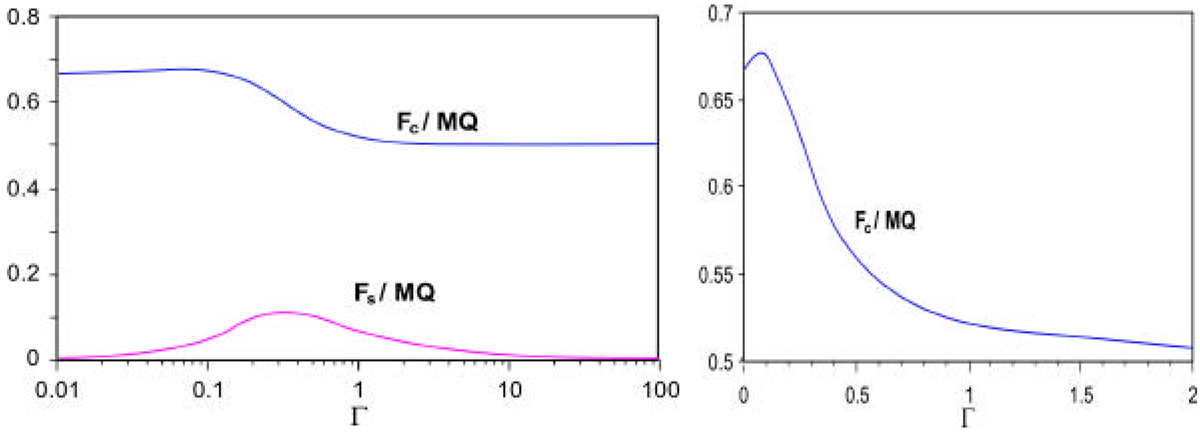

Figure 6. The waves amplitude versus Γ.

The dependence of the waves amplitude on the parameter Γ is given in Fig. 6. The main (cos-like) wave amplitude distribution for small Γ is presented separately. The sin-like wave amplitude reaches the maximum at $\Gamma \approx 0.4$ and vanishes for small and high Γ. For $\Gamma = 0$, the cos-like item gives the resonator term of monopole longitudinal wake potential for dc case [2]:

$$W_z^R(s)_{\Gamma\to 0} = -16/3b^2 e^{-s/s_0} \cos\left(\sqrt{3}s/s_0\right). \qquad (31)$$

Note, that the AC wake potential resonator term amplitude (29) contains both the contribution from cos and sin-like waves and the retarding potential seen by the driving charge has non-zero phase.

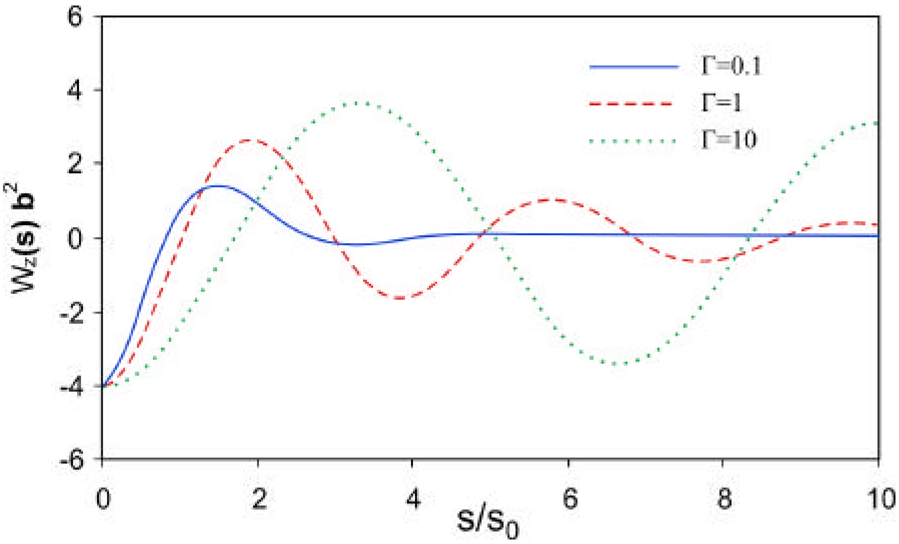

Figure 7. Longitudinal monopole wake potential for AC conductivity case.

To obtain the diffusion part of the longitudinal wake potential, we note, that during the integration along the brunch cut (Fig.5) the values of parameter l (3) are the same at both cut off sides when $\mathrm{Im}(k) < -1/\Gamma$ and the integral along this part of cut off is equal to zero. The diffusion part of the longitudinal wake potential is then given by the integration over k along of both sides of cut from $k=0$ to $k=-i/\Gamma$ (Fig.5) and is given by:

$$W_z^D(s) = \frac{16\sqrt{2}}{b^2 p} \int_0^\infty \frac{x^2(1+x^2\Gamma)}{8\cdot(1+x^2\Gamma)^4 + x^6} e^{-\frac{s}{s_0}\frac{x^2}{1+x^2\Gamma}} dx \qquad (32)$$

that tends to the corresponding diffusion term of wake function for dc conductivity if $\Gamma \to 0$. The complete wake function is a sum of resonator (28) and diffusion (32) terms:

$$W_z(s) = W_z^R(s) + W_z^D(s) \qquad (33)$$

Figure 7 presents the point charge monopole longitudinal wake potential for various Γ. Note, that for both limiting cases, ($\Gamma \to 0$; $\Gamma \to \infty$), the results are strive to well known performance of the wake potentials [2].

4. SUMMARY

The analytical presentations for the wake potentials produced by point and Gaussian bunches are obtained for the AC and DC conductivity of the pipe walls.

For DC conductivity case, the integral presentation of the point wake potential is transformed to analytical form for any multiple modes of both longitudinal and transverse wake potentials. An obtain results are further expanded into fast converged series that gives pictorial view of excited short range wakes and allow to evaluate the correction terms with respect to existing approximation (see Piwinski formula [6]).

For AC case, the inverse Fourier transformation of the longitudinal impedance is obtained in analytical form. The resonator term of the longitudinal wake potential is derived in the form of superposition of two waves shifted by $p/2$. The amplitude of the resonator term is defined by contribution of both waves and is greater than in single mode approximation (see [2]). The diffusion term is presented in simple integral form.

5. REFERENCES

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