# Computation of the Short and Long Range Wake in Periodic Circular Symmetric Accelerating Structures Using the Field Matching Technique

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## Abstract

**In this contribution, the field matching technique is employed for the calculation of the short and long range wake in periodic circular symmetric accelerating structures. The electromagnetic field in the beampipe is represented by a superposition of spatial harmonics; and the cavity field is expanded in terms of functions which already fulfill the boundary conditions at the cavity walls. The numerical results show that the beam impedance of such structures behaves like a reactance function with alternating poles and zeros. Two methods which are appropriate for the calculation of the wake function from this kind of beam impedance are discussed. The short and the long range wake is computed for various cases. Furthermore the loss parameters are compared with those of a previously published approach which can be used in order to calculate the short range wake for short bunches. Finally the loss parameters of a simple model of the TESLA accelerating structure are computed and compared with those numbers which are given in the TESLA conceptual design report.**

# I. Introduction

The field matching method was recently used for the computation of the beam parameters of periodic structures [1]-[5]. Such structures are encountered very often in the field of linear accelerators: E.g., the may be used to study the effect of surface roughness in long beampipe sections, which is especially interesting for the FEL operation mode of TESLA [6], [7] as well as to calculate the wakefields in the accelerating structure itself.

The planar configuration shown in Fig. 1 served in [3] as a model for a circular accelerating structure. In [5] it was demonstrated that the beam impedance of this model and that of the corresponding circular structure, which is presented in Fig. 2(a), are indeed very similar except for very low frequencies [8]. The dc spectral component of the beam impedance of the circular structure in fact vanishes; whereas this is not the case for planar gratings. It was thus found out in that paper that the wake function and the loss parameter of circular structures are quite different from those corresponding to planar gratings for long bunches which basically excite the low frequency part of the wakefield. On the other, the numerical investigations presented in [5] showed that if one is interested in the short range wake for short bunches only one may use the planar grating model.

The structure which was investigated in [5] radiates electromagnetic energy through the gaps between the diaphragms which is also indicated in Fig. 2(a). We have therefore to close

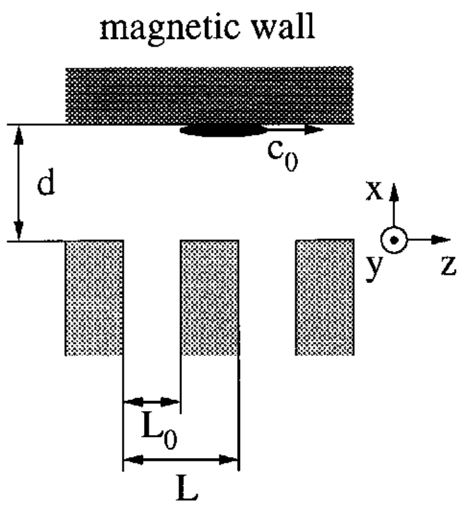


Figure 1: Planar model for an accelerating structure.

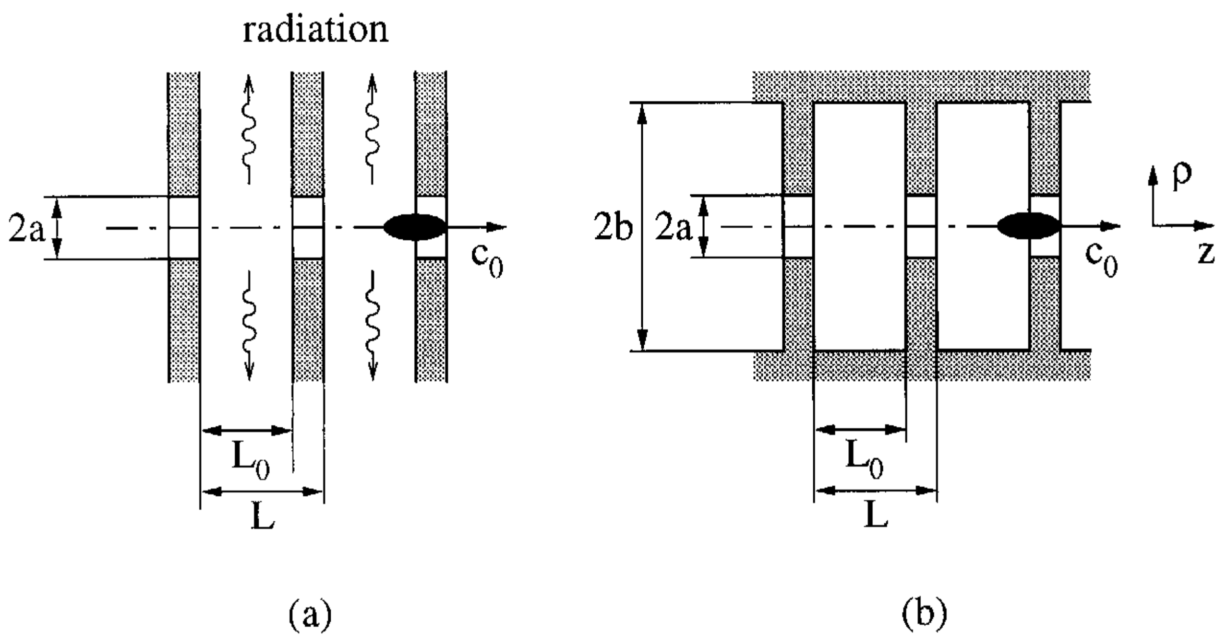


Figure 2: (a) Previously used model for the computation of the short range wake. (b) Actual accelerating structure.

these gaps as it is the case for a real accelerating structure, see Fig. 2(b), if we intend to compute also long range wake effects.

This contribution is dedicated to the computation of the short and long range wake effects in structures like that which is presented in Fig. 2(b). The field matching techniques corresponding to the open and the closed structure are very similar. We just have to substitute some Hankel functions which describe radially outward travelling waves between the diaphragms by linear combinations of Bessel and Hankel functions yielding a radial standing wave pattern which already fulfills the boundary condition at $\rho = b$.

The open model is damped by energy radiation. This leads to a complex-valued beam impedance without poles along the frequency axis. On the other hand, the closed structure is lossless because the electromagnetic energy is trapped within the shielding which is assumed to be perfectly conducting. The beam impedance of this structure is therefore purely imaginary and behaves like a reactance function with alternating poles and zeros.

This type of beam impedance requires some special techniques for the calculation of the wake function. Two methods will be discussed which solve the problem: The first one is based on the computation of the $\delta$-wake from the resonant frequencies of the poles and their weighting factors. The actual wake for a certain bunch is then obtained from the corresponding convolution integral.

The second methods already takes into account the bunch shape in the frequency domain which means that the beam impedance is multiplied by the bunch spectrum and then inverse Fourier transformed in order to determine the wake function. Similar to the first method, the poles are treated analytically; whereas the remaining part of the spectrum which is well-behaved is numerically inverse Fourier transformed.

Numerical results for the short and the long range wake will be presented for an excitation of the accelerating structure by various bunches. It will be demonstrated that more and more poles are significantly excited if the bunch length is reduced. It will also be shown that the short range wake and the loss parameter corresponding to the open and the closed circular structure are approximately the same for very short bunches.

Moreover a simple model of the TESLA accelerating structure will be considered. This model does not take into account that the contour of the TESLA structure changes smoothly. It rather approximates the cells by abrupt steps in the radius according to Fig. 2(b). The corresponding loss parameter will be given as a function of the bunch length and will be compared with those numbers which are given in the conceptual design report (CDR) for TESLA [6]. It will turn out that really a large number of poles has to be taken into account if accurate results are required for short bunches.

# II. Theory

This section is organized as follows: In Subsection IIa) it is shown how the previously published field matching technique for the open structure has to be modified for the analysis of the closed structure presented in Fig. 2(b). Subsection IIb) is then dedicated to the computation of the wakefield from a beam impedance which looks like a reactance function.

## IIa) Field analysis

The field analysis of the accelerating structure which is shown in Fig. 2(b) is very similar to that of the previously used model for the computation of the short range wake which is presented in

Fig. 2(a). In the latter structure, the electromagnetic field is given by the eigenmode expansion

$$\widetilde{H}_\phi^{(2)} = \sum_{n=0}^{\infty} \mathcal{A}_n \cos(k_{zn} z)\, \mathrm{H}_1^{(2)}\left(k_{\rho n}^{(2)} \rho\right) \quad , \tag{1}$$

$$\widetilde{E}_z^{(2)} = Z_0 \sum_{n=0}^{\infty} \mathcal{A}_n \frac{k_{\rho n}^{(2)}}{j k_0} \cos(k_{zn} z)\, \mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)} \rho\right) \quad , \tag{2}$$

for $\rho \geq a$ and $0 \leq z \leq L_0$, where $\mathrm{H}_0^{(2)}$ and $\mathrm{H}_1^{(2)}$ are the zeroth and first order Hankel functions of second kind. The Hankel functions describe radially outward propagating waves which can be seen from the asymptotic behaviour of these functions [9]:

$$\mathrm{H}_\nu^{(2)}(k\rho) \approx \sqrt{\frac{2j}{\pi k \rho}}\, j^\nu e^{-jk\rho} \quad \text{for} \quad |k\rho|^2 \gg \left|\nu^2 - \frac{1}{4}\right| \tag{3}$$

On the other hand, the cavity wall at $\rho = b$ in an actual accelerating structure requires

$$\widetilde{E}_z^{(2)}(\rho)\Big|_{\rho=b} = 0 \quad . \tag{4}$$

This boundary condition is fulfilled if we substitute the Hankel functions $\mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)}\rho\right)$ and $\mathrm{H}_1^{(2)}\left(k_{\rho n}^{(2)}\rho\right)$ by appropriate linear combinations of Bessel and Hankel functions $\mathrm{Z}_0\left(k_{\rho n}^{(2)}\rho\right)$ and $\mathrm{Z}_1\left(k_{\rho n}^{(2)}\rho\right)$, respectively:

$$\mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)}\rho\right) \mapsto \mathrm{Z}_0\left(k_{\rho n}^{(2)}\rho\right) = \mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)}\rho\right) \mathrm{J}_0\left(k_{\rho n}^{(2)}b\right) - \mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)}b\right) \mathrm{J}_0\left(k_{\rho n}^{(2)}\rho\right) \quad , \tag{5}$$

$$\mathrm{H}_1^{(2)}\left(k_{\rho n}^{(2)}\rho\right) \mapsto \mathrm{Z}_1\left(k_{\rho n}^{(2)}\rho\right) = \mathrm{H}_1^{(2)}\left(k_{\rho n}^{(2)}\rho\right) \mathrm{J}_0\left(k_{\rho n}^{(2)}b\right) - \mathrm{H}_0^{(2)}\left(k_{\rho n}^{(2)}b\right) \mathrm{J}_1\left(k_{\rho n}^{(2)}\rho\right) \tag{6}$$

Note that

$$\mathrm{Z}_0\left(k_{\rho n}^{(2)}b\right) = 0 \tag{7}$$

so that Eq. (4) is satisfied. It is worth noting that the substitutions according to Eqs. (5) and (6) are sufficient in order to analyze the closed structure instead of the open one. The already implemented computer code for the open structure remains therefore almost the same.

## IIb) Calculation of the wakefield

In [5] it has been shown that the beam impedance per unit length is proportional to the expansion coefficient of the zeroth order spatial harmonic divided by the wavenumber. This beam parameter is a complex-valued, well-behaved function of frequency for open structures. Fig. 3 presents the real and imaginary part of $Z'(k_0)\,/\,(Z_0/L)$ for a typical set of parameters.

On the other hand, closed structures are not damped by radiation losses. Hence the corresponding $Z'(k_0)$ is purely imaginary. Fig. 4 shows a typical $Z'(k_0)$. It is characterized by alternating poles and zeros and looks thus like a reactance function. Such a function is given by a series of terms $jk_0/\left((jk_0)^2 - (jk_{0i})^2\right)$ where $k_{0i}$ denotes the resonance wavenumber of the $i$th pole. Note that although the numerical results suggest that $Z'(k_0)$ is a reactance function it has actually not yet been proven. At least it is clear that $Z'(k_0)$ cannot be a superposition of terms $jk_0/\left((jk_0)^2 - (jk_{0i})^2\right)$ with complex-valued $k_{0i}$ because this would contradict the fact that $Z'(k_0)$ is purely imaginary for all frequencies.

Nevertheless, let us assume in this paper that $Z'(k_0)$ is a reactance function. Then the question arises how can we calculate the wake function from this kind of beam impedance. In the following two methods are presented which solve this problem.

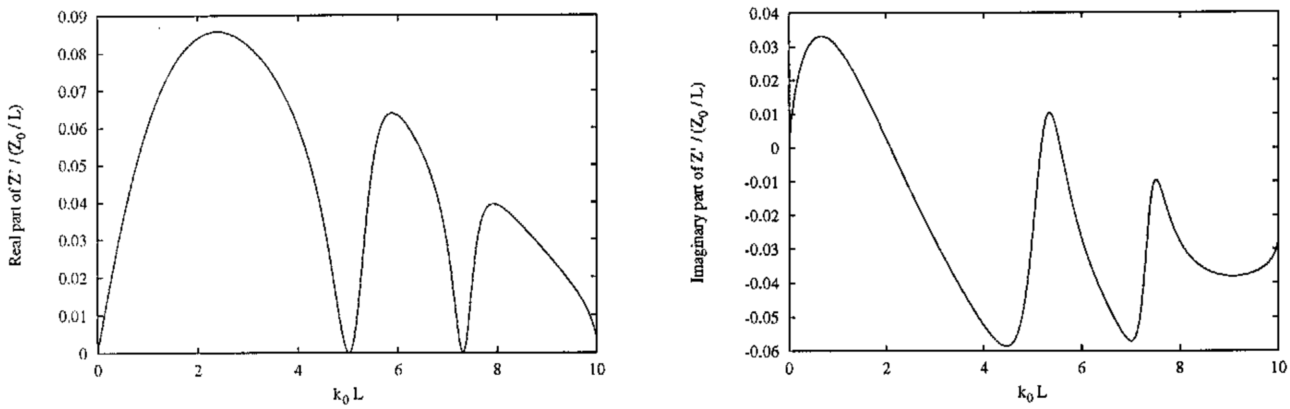


Figure 3: Real and imaginary part of the beam impedance per unit length as a function of frequency for the open structure. Parameters: $L_0 = 0.3L$, $a = 0.5L$ and $N_{har} = 50$.

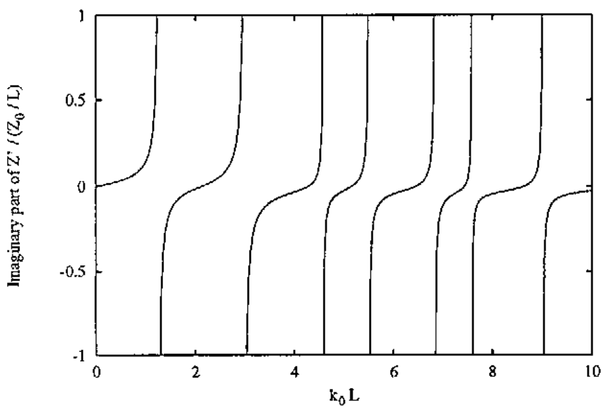


Figure 4: Imaginary part of the beam impedance per unit length as a function of frequency for the closed structure. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$ and $N_{har} = 50$.

### IIb.1) Convolution method

The method which is discussed first is based on the computation of the $\delta$-wake. The actual wake of a real bunch is then calculated by convoluting the bunch shape and the $\delta$-wake. This method is thus referred to as the *convolution method.*

If the assumption that the beam impedance per unit length is a reactance function is correct, it then can be written as [10]

$$Z'(k_0) = \sum_{i=1}^{\infty} r_i \frac{j\dfrac{k_0}{k_{0i}}}{1-\left(\dfrac{k_0}{k_{0i}}\right)^2} \quad , \tag{8}$$

where the $i$th pole is weighted by the factor $r_i$. The weight factor $r_i$ can readily be computed numerically from the slope of $1/Z'(k_0)$ at $k_0 = k_{0i}$:

$$\frac{1}{r_i} = -\frac{jk_{0i}}{2} \left. \frac{d\left(\dfrac{1}{Z'(k_0)}\right)}{dk_0} \right|_{k_0=k_{0i}} \tag{9}$$

Note that $1/Z'(k_0)$ vanishes at $k_0 = k_{0i}$.

The $\delta$-wake per unit length reads

$$W^{\delta\prime}(s) = \frac{c_0}{2\pi} \int_{k_0=-\infty}^{\infty} Z'(k_0)\, e^{jk_0 s} dk_0 \quad . \tag{10}$$

Hence $W^{\delta\prime}(s)$ can be calculated if we substitute Eq. (8) into Eq. (10). The evaluation of the infinite integral in Eq. (10) so that $W^{\delta\prime}(s)$ is causal is discussed in detail in the Appendix yielding

$$W^{\delta\prime}(s) = c_0 \sum_{i=1}^{\infty} r_i k_{0i} \cos(k_{0i} s) \quad . \tag{11}$$

The actual wake per unit length for a given bunch shape $\lambda(\xi)$ can then be obtained using the well-known convolution formula

$$W'(s) = \int_{\xi=-\infty}^{\infty} \lambda(\xi)\, W^{\delta\prime}(s+\xi)\, d\xi \quad . \tag{12}$$

It is worth noting that the wake is completely determined by the poles of $Z'(k_0)$. This means that we do not have to know $Z'(k_0)$ precisely for the numerical evaluation of the convolution method; we just need to compute the resonant wavenumbers $k_{0i}$ and the corresponding weighting factors $r_i$. Nevertheless we have to keep in mind that this statement is only valid under the assumption that $Z'(k_0)$ can be written according to Eq. (8) which has yet not been proven.

### IIb.2) FFT method

The second method for the computation of the wake is called *FFT method* because it requires a FFT instead of a convolution integral. The FFT method is based on the fact that the wake is just the inverse Fourier transformation of the beam impedance times the bunch spectrum:

$$W'(s) = \frac{c_0}{2\pi} \int_{k_0=-\infty}^{\infty} Z'(k_0)\, \tilde{\lambda}(k_0)\, e^{jk_0 s} dk_0 \tag{13}$$

The function $Z'(k_0)\,\widetilde{\lambda}(k_0)$ still contains poles at the same wavenumbers $k_{0i}$ as the beam impedance does. If we substract these poles which are weighted by $\widetilde{\lambda}(k_{0i})\,r_i$ from $Z'(k_0)\,\widetilde{\lambda}(k_0)$ we get a well-behaved spectral distribution $S(k_0)$

$$S(k_0) = Z'(k_0)\,\widetilde{\lambda}(k_0) - \sum_{i=1}^{\infty} \widetilde{\lambda}(k_{0i})\,r_i \frac{j\dfrac{k_0}{k_{0i}}}{1-\left(\dfrac{k_0}{k_{0i}}\right)^2} \quad . \tag{14}$$

Fig. 5 presents this auxiliary function with the bunch length as a parameter. Note that $S(k_0)$

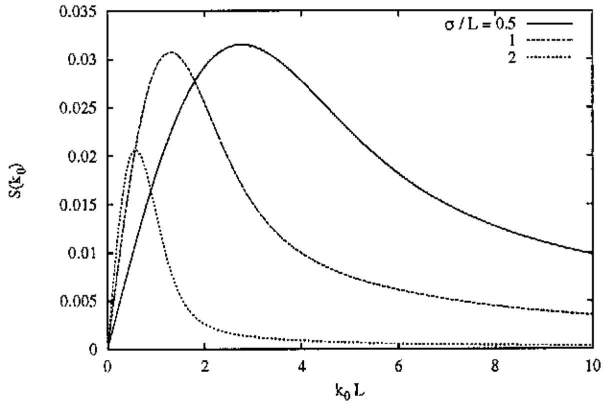


Figure 5: Auxiliary function for the wake calculation with the bunch length as a parameter. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$ and $N_{har} = 50$.

is absolutely pole-free. Furthermore Fig. 5 shows that the spectral width of $S(k_0)$ increases if we consider shorter bunches which is also expected.

The subtracted poles and the auxiliary function $S(k_0)$ are then individually inverse Fourier transformed:

$$W'(s) = \underbrace{c_0 \sum_{i=1}^{\infty} \widetilde{\lambda}(k_{0i})\,r_i k_{0i} \cos(k_{0i}s)}_{\text{pole part}} + \underbrace{\frac{c_0}{2\pi}\int_{k_0=-\infty}^{\infty} S(k_0)\,e^{jk_0 s}\,dk_0}_{\text{FFT part}} \tag{15}$$

The pole part of $W'(s)$ is obtained by inverse Fourier transforming the sum in Eq. (14) analytically similar to the convolution method. As already mentioned, the weighting factors of the poles are now $\widetilde{\lambda}(k_{0i})\,r_i$ instead of $r_i$. Therefore only a few resonances contribute significantly to this summation as long as the bunch has a reasonable length.

Fig. 6 presents both the FFT and the pole part of the FFT method. The pole part has a step discontinuity at $s = 0$ which is $c_0 \sum_{i=1}^{\infty} \widetilde{\lambda}(k_{0i})\,r_i k_{0i}$. This discontinuity is just compensated by the inverse Fourier transformation of $S(k_0)$ so that the total wake is continuous at $s = 0$. The fast oscillations of the FFT part are due to the fact that the spectral distribution $S(k_0)$ has to be truncated somewhere. This phenomenon could be removed by an asymptotic expansion of $S(k_0)$.

Fig. 7 compares the total wake corresponding to the FFT and the convolution method. The agreement between both wake functions is excellent except for the already discussed oscillations of the FFT method at $s = 0$ which are an artefact of the numerical calculations. Nevertheless

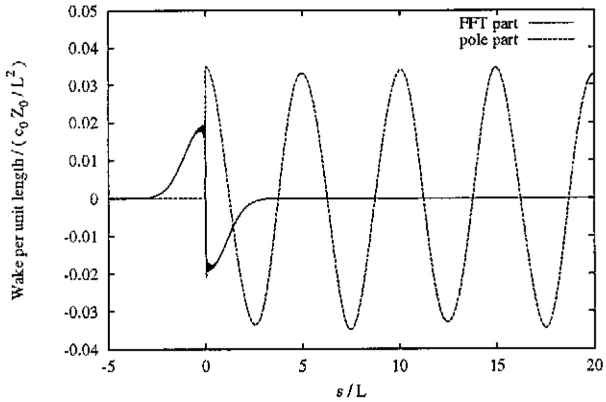


Figure 6: FFT and pole part of the FFT method. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$, $k_{0m} = 100/L$, $N_{har} = 50$ and $\sigma = L$.

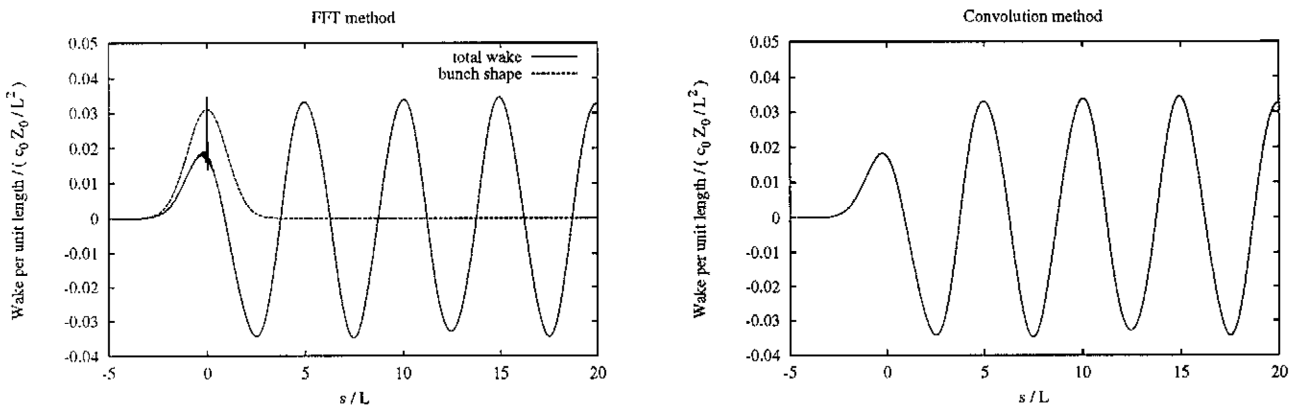


Figure 7: Comparison between the FFT and the convolution method. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$, $k_{0m} = 100/L$, $N_{har} = 50$ and $\sigma = L$.

we are going to use only the convolution method for the following calculations of the wake function because this method does not suffer from such a phenomenon.

# III. Numerical results

This section in which some numerical results are presented is split into two subsection. The first one is dedicated to the verification of the developed method; whereas the computation of the loss parameter of a simple model for the TESLA accelerating structure is discussed in subsection IIIb).

## IIIa) Verification of the presented method

In order to check the validity of the presented method let us compare the cavity and the waveguide field at $\rho = a$. Fig. 8 shows the corresponding real and imaginary parts of the

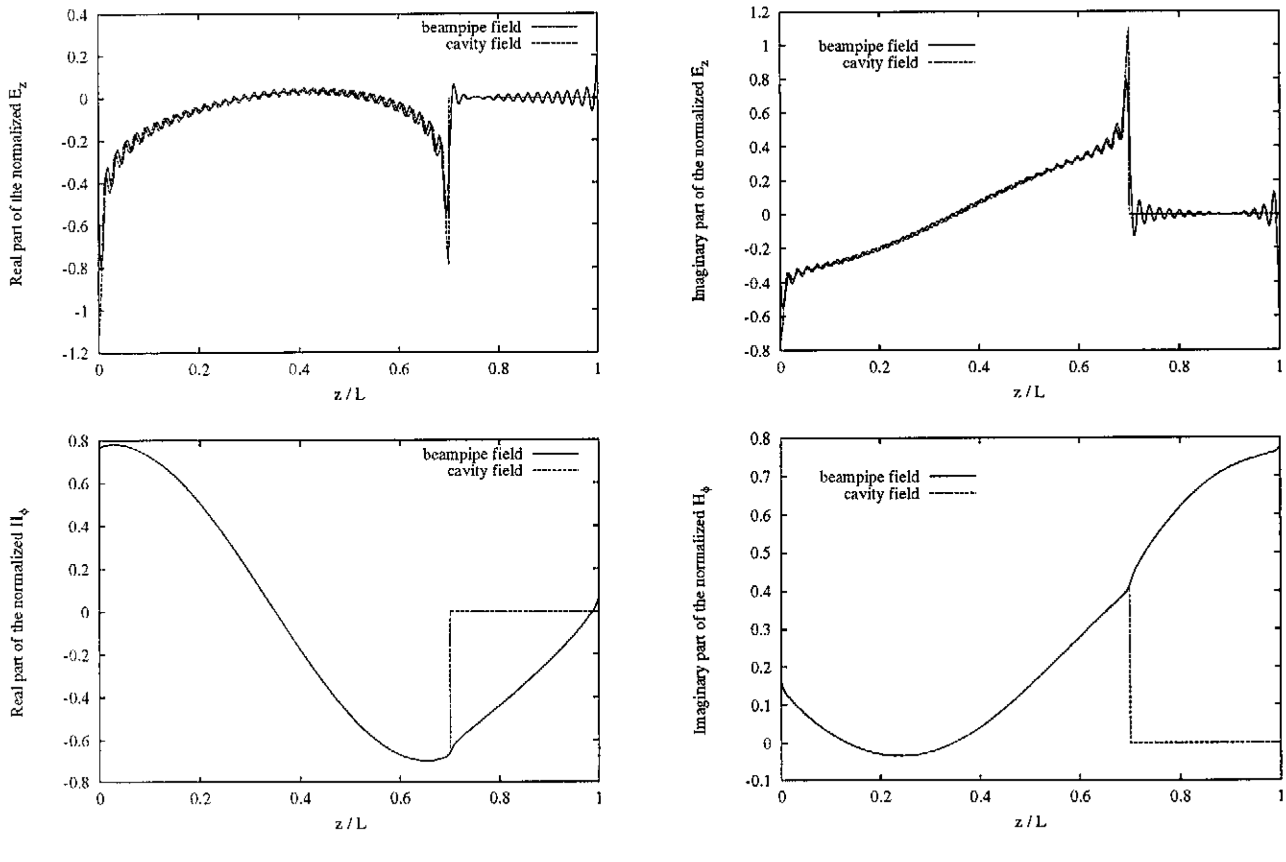


Figure 8: Normalized axial electric und azimuthal magnetic field along $\rho = a$. Parameters: $L_0 = 0.7L$, $a = 0.5L$, $b = L$, $k_0 = 5/L$ and $N_{har} = 50$.

axial electric and the azimuthal magnetic field. The wavenumber $k_0 = 5/L$ corresponds to a frequency of the electromagnetic field of about 2.5 GHz if we assume a period lenght $L$ of 10 cm; and $N_{har} = 50$ means that 50 spatial harmonics are taken into account in the beampipe.

The axial electric field shows the well-known singularities at $z = 0$ and $z = L_0$ due to the 90°-edges at these locations [11]. This phenomenon leads to relative strong oscillations of the field distributions. Nevertheless, the agreement between the beampipe and the cavity field is quite good for $0 < z < L_0$. On the other hand, for $L_0 < z < L$, the cavity field is not defined and the beampipe field must vanish as it is the case.

The azimuthal magnetic field is parallel to the edges. This field component is consequently well-behaved everywhere. The corresponding curves for the beampipe and the cavity field are very smooth and agree almost perfectly within $0 \leq z \leq L_0$.

Fig. 9 presents the wake functions for various bunches. The parmeter $k_{0m} = 100/L$ means

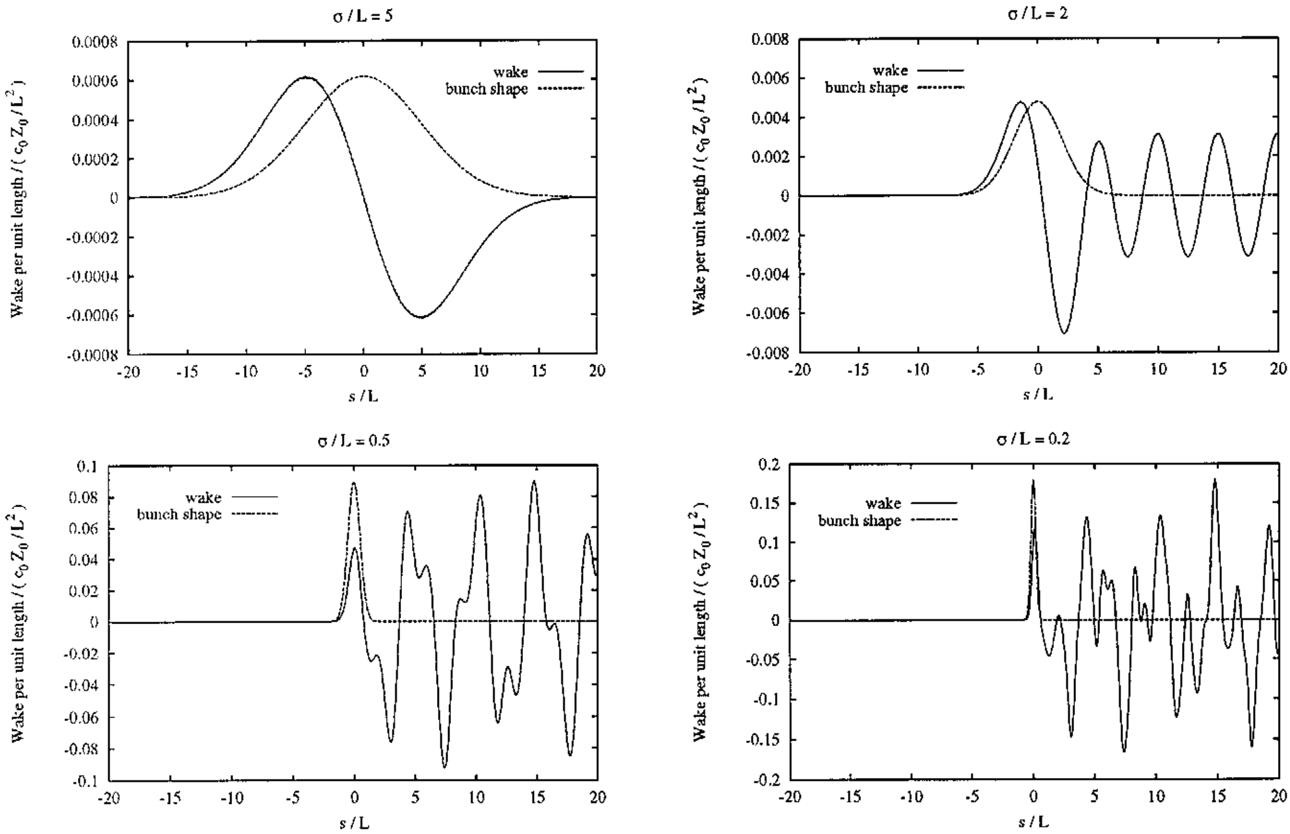


Figure 9: Wake as a function of the bunch length. Parameters: $L_0 = 0.7L$, $a = 0.5L$, $b = L$, $k_{0m} = 100/L$ and $N_{har} = 50$.

that all resonances up to this wavenumber are taken into account. For a period length of 10 cm, $k_{0m} = 100/L$ corresponds to a maximum frequency of about 50 GHz. Within this frequency band 638 poles are found.

For a very long bunch ($\sigma/L = 5$) the wake is inductive. This means that $W'(s)$ is proportional to the derivative $d/ds\lambda(s)$. For such a long bunch none of the poles of $Z'(k_0)$ is significantly excited because the bandwidth of $\tilde{\lambda}(k_{0i})$ is less than the lowest resonance wavenumber $k_{01}$. Therefore no long range wake is observed in this case.

If we consider a bunch with $\sigma/L = 2$ the short range wake is still almost inductive. Nevertheless, the lowest pole is considerably excited, whereas the excitation of higher order resonances is negligible. The long range wake is therefore a harmonic function with a wavelength equal to $2\pi/k_{01}$.

If we consider shorter bunches ($\sigma/L = 0.5$ and $\sigma/L = 0.2$) the excitation of more and more poles cannot be neglected any longer. Under these circumstances many resonances contribute significantly to the long range wake according to the pole part of Eq. (15).

On the other hand, the short range wake finally becomes almost capacitive ($W'(s) \propto \int_s \lambda(s)\, ds$) for very short bunches as it is shown in Fig. 10. In this figure $\sigma/L = 0.02$ is as-

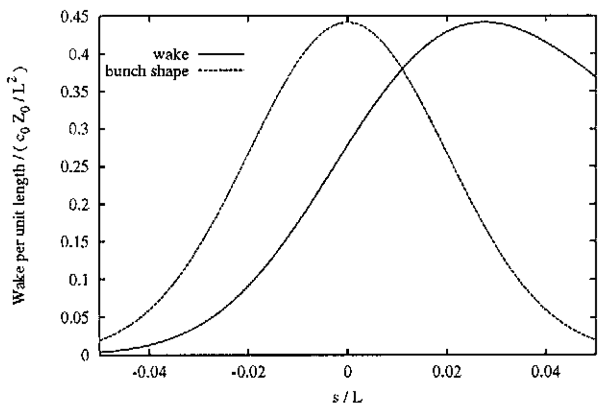


Figure 10: Capacitive short range wake for a very short bunch length. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$, $k_{0m} = 100/L$, $N_{har} = 50$ and $\sigma = 0.02L$.

sumed which means that the actual bunch length is 2 mm for $L = 10$ cm.

For a further check of the presented method let us compare the short range wake which is calculated using this method with that obtained for the open structure (Fig. 2(a)). Fig. 11 illustrates that both wakes are the same for

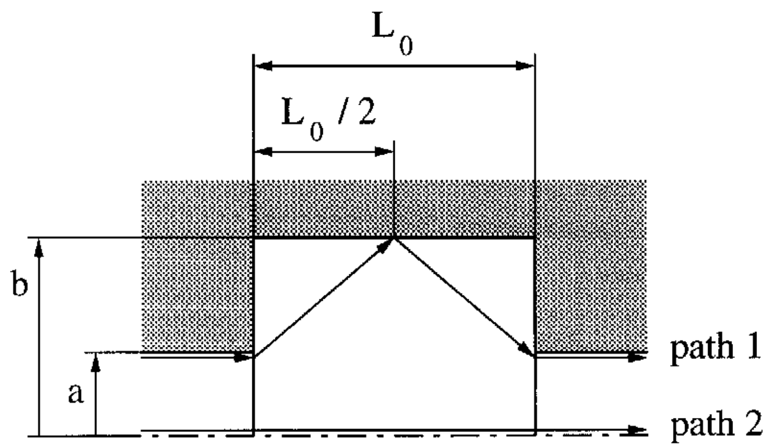


Figure 11: Shortest path of a wave reflection (path 1) along which a bunch (moving along path 2) can experience its own wakefield due to the outer shielding of the cavity.

$$\Delta s < L_0 \left( \sqrt{1 + 4\left(\frac{b-a}{L_0}\right)^2} - 1 \right) \quad , \tag{16}$$

where $\Delta s$ denotes the difference between the lenghts corresponding to path 1 and 2. Path 1 is the shortest path of a wave reflection along which a bunch can experience its own wakefield due to the outer shielding of the cavity; while the bunch itself moves along path 2. Note that the interval $\Delta s$ counts from that coordinate $s$ where the bunch first gives rise to a non-neglible wake.

Fig. 12 presents the wake functions corresponding to the open and the closed structure for

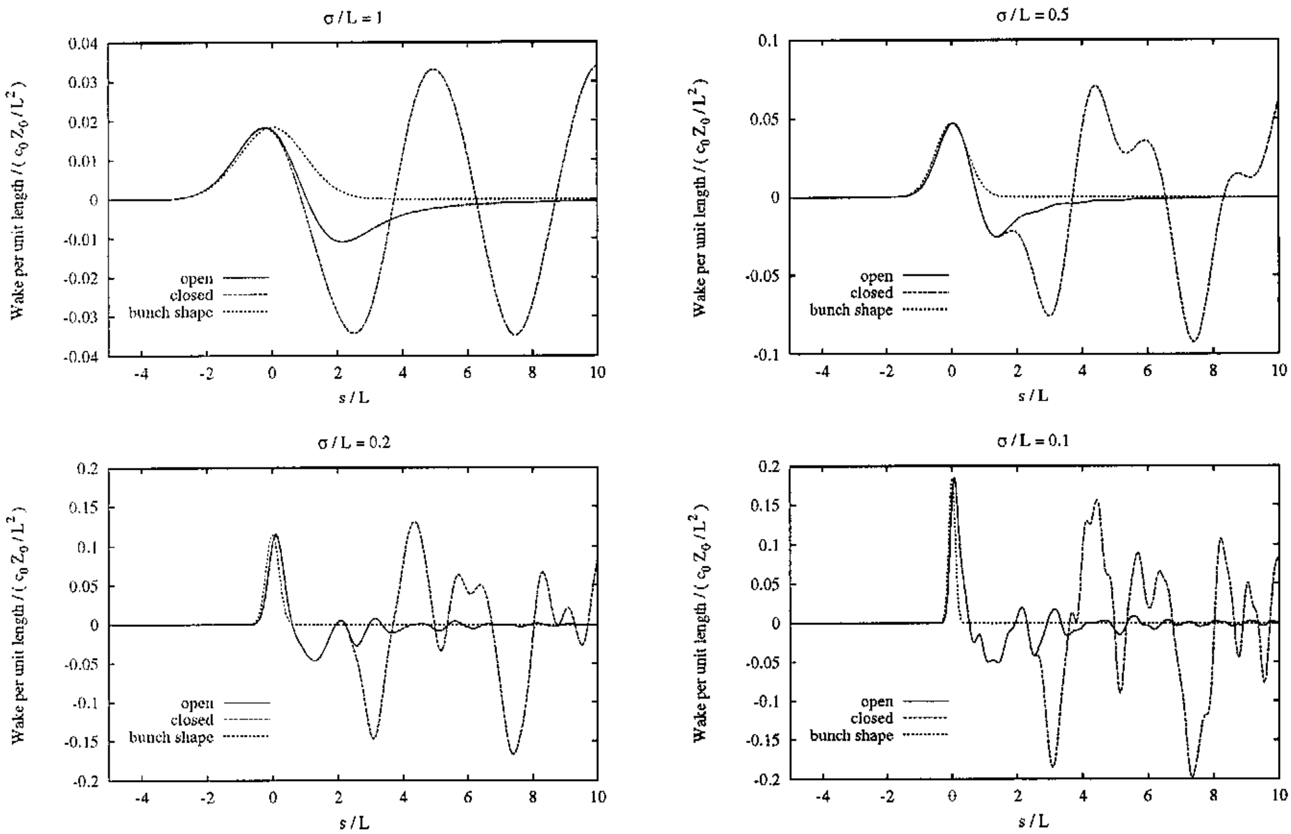


Figure 12: Comparison between the wakes which correspond to the open and the closed structure for various bunches. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$, $k_{0m} = 100/L$ and $N_{har} = 50$.

various bunches. For the given parameters we have $\Delta s = 2.7L$. The presented curves confirm that the wake functions are indeed identical within this interval $\Delta s$ in all four considered cases.

For a long bunch ($\sigma/L = 1$) the wake functions already start to deviate within the bunch. We therefore expect that both models do not yield the same loss parameter for such an excitation. On the other hand the short range wakes agree very well for ($\sigma/L \leq 0.5$).

This statement is also confirmed by Fig. 13 which compares the loss parameter of both models as a function of the bunch length. From the numerical results which are presented in this figure it clearly follows that the models yield the same results for short bunches. This justifies the use of an open structure as a model for a real accelerating structure if one is interested in the short range wake only and if the exciting bunch fulfills the condition $\sigma \ll \Delta s$.

## IIIb) A simple model for the TESLA accelerating structure

In this subsection some of the problems are discussed which one encounters if one is interested in the beam parameters for very short bunches. The parameter set which is assumed for this

| $\sigma$ in mm | $k'$ in V / pC / m | |
|---|---|---|
| | CDR | this method |
| 0.5 | 12.0 | 10.8 |
| 1.0 | 9.7 | 9.2 |

Table 1: Comparison of the loss parameter corresponding to two short bunches with those given in the CDR. Parameters: $L_0 = 0.792L$, $a = 0.303L$, $b = 0.895L$ and $N_{har} = 100$.

discussion corresponds to a simple model for the TESLA accelerating structure ($L_0 = 0.792L$, $a = 0.303L$ and $b = 0.895L$).

For the calculations 100 spatial harmonics were taken into account which is quite a lot. Using this number of field expansion functions in the beampipe, we can expect accurate results up to a maximum normalized wavenumber of $k_0 L \approx 250$. Keeping in mind that the length of one TESLA cell is 115.4 mm, $k_0 L = 250$ corresponds to a frequency of about 100 GHz. This upper frequency limit is found from a detailed study of convergence for the beam impedance.

Fig. 14 presents the loss parameter of the TESLA accelerating structure as a function of the bunch length with the wavenumber up to which poles are taken into account as a parameter. From the curves which are given in this figure one can see that $k_{0m}L = 10$ is sufficient for $\sigma/L = 0.1$; whereas we have to consider all poles up to $k_{0m}L = 200$ for $\sigma/L = 0.005$. The wavenumber up to which the resonances have to be determined in order to get accurate results for the loss parameter is actually inverse proportional to the bunch length.

Figs. 15 and 16 present the weighting factors $r_i$ according to Eq. (9) and the number of poles as a function of frequency, respectively. Fig. 15 shows that the mean value of the weighting factors $r_i$ decreases with increasing wavenumber. On the other hand, the number of poles is proportional to $k_0^2$ which can be seen from the two curves that are given in Fig. 16.

From Fig. 14 it follows that the presented method still yields accurate loss parameters for a 500 $\mu$m short bunch if $k_{0m}L = 200$ is used. In this case more than 1600 poles have to be taken into account. The numerical search for all these poles requires more than 20 days of cpu-time on a modern workstation. The efficiency of the field matching method has therefore significantly to be improved if one is interested in the beam parameters of even shorter bunches like those which are foreseen for the TESLA FEL operation mode.

In Table 1, the loss parameters corresponding to two short bunches are compared with those values which are given in the CDR [6], [12] for the average loss parameter of the 8th cavity. This cavity and the first cell are more than 7 m apart which is much more than the critical length for a periodic solution

$$L_{crit} = \frac{a^2}{2\sigma} \tag{17}$$

even for $\sigma = 0.5$ mm. This length actually is 600 mm and 1200 mm for $\sigma = 0.5$ mm and $\sigma = 1.0$ mm, respectively, assuming $a = 35$ mm.

The loss parameters which are presented in the CDR and those which are calculated using the field matching method agree quite well if we take into account that only a rough model of the TESLA accelerating structure is used. The fact that the loss parameters in the CDR are slightly higher than ours may be explained by two reasons: It is known that the loss parameters of a structure with a finite number of cells are larger than those of a corresponding infinite structure. Furthermore we might have overlooked some of the poles which leads to a degradation of the calculated loss parameters.

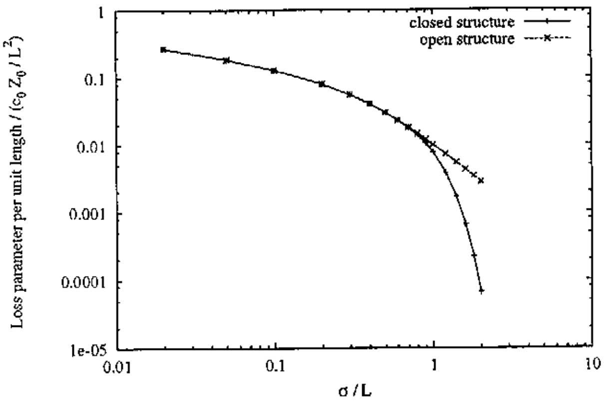


Figure 13: Comparison between the loss parameters of a closed structure and the corresponding open model as a function of the bunch length. Parameters: $L_0 = 0.3L$, $a = 0.5L$, $b = 2L$, $k_{0m} = 100/L$ and $N_{har} = 50$.

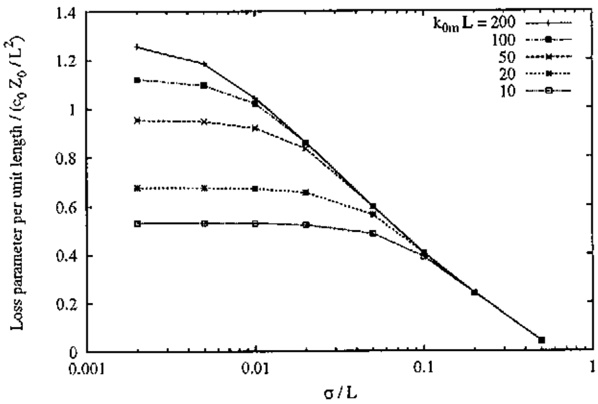


Figure 14: Loss parameter per unit length of the TESLA accelerating structure as a function of the bunch length with the wavenumber up to which poles are taken into account as a parameter. Parameters: $L_0 = 0.792L$, $a = 0.303L$, $b = 0.895L$ and $N_{har} = 100$.

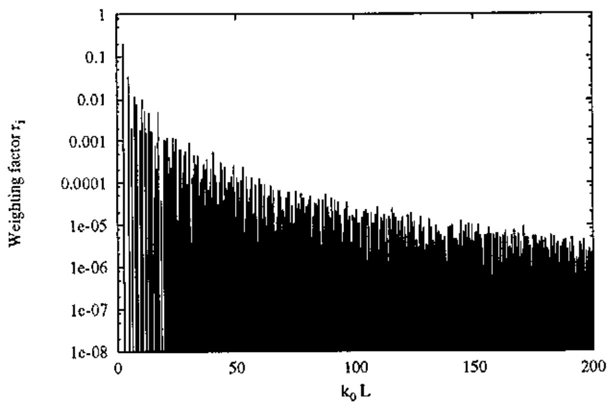


Figure 15: Weighting factors $r_i$ of the TESLA structure as a function of frequency. Parameters: $L_0 = 0.792L$, $a = 0.303L$, $b = 0.895L$ and $N_{har} = 100$.

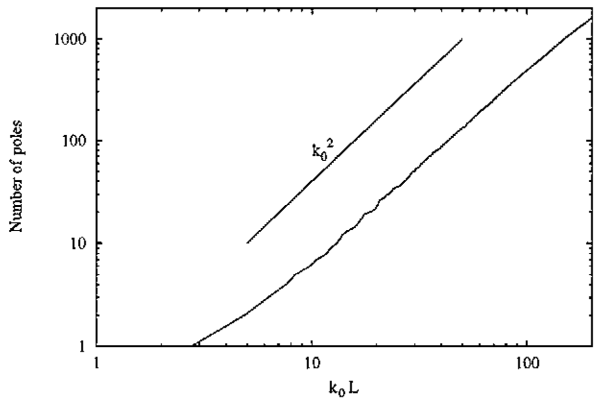


Figure 16: Number of poles as a function of frequency. Parameters: $L_0 = 0.792L$, $a = 0.303L$, $b = 0.895L$ and $N_{har} = 100$.

# IV. Conclusions

The field matching technique has been applied for the computation of the electromagnetic field and the beam parameters of periodic circular symmetric accelerating structures which are excited by an ultra-relativistic point charge moving along the longitudinal axis of the structure. It has been demonstrated that the beam impedance of such structures looks like a reactance function. It has been shown how the wake function can be computed from this kind of beam impedance. Numerical results have been presented for the beam parameters of various structures. Especially the loss parameter of a simple model of the TESLA accelerating structure has been calculated for short bunches.

# Acknowledgement

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# References

[1] M. Dohlus, N. Holtkamp and A. Jöstingmeier, "Design of a broadband absorber for $f >$ 100 GHz", *Meeting note: 26 Linear collider project meeting at DESY*, 1997.

[2] M. Dohlus, N. Holtkamp, A. Jöstingmeier, H. Hartwig and D. Trines, "Design of a HOM broadband absorber for TESLA", *Meeting note: 31 Linear collider project meeting at DESY*, 1998.

[3] A. Jöstingmeier, M. Dohlus and N. Holtkamp, "Application of the mode matching technique for the computation of the beam parameters of an infinite periodic structure", DESY, TESLA 98-23, 1998.

[4] A. Jöstingmeier, M. Dohlus and N. Holtkamp, "Computation of the absorption characteristics of a two-dimensional rectangular waveguide array using the mode matching technique", DESY, TESLA 98-24, 1998.

[5] A. Jöstingmeier and M. Dohlus, "Comparison of the beam parameters corresponding to corrugated circular waveguides and planar gratings", DESY, TESLA 98-30, 1998.

[6] R. Brinkmann *et al.* (ed.), *Conceptual design of a 500 GeV* $e^+e^-$ *linear collider with integrated X-ray laser facility*, DESY 1997-048, 1997.

[7] R. Brinkmann, "On the free electron laser mode operation in TESLA", DESY, TESLA 96-01, 1996.

[8] H. Henke, "Point charge passing a resonator with beam tubes", *Archiv für Elektrotechnik*, vol. 69, pp. 271–277, 1986.

[9] H.-G. Unger, *Elektromagnetische Theorie für die Hochfrequenztechnik*, Hüthig, 1981.

[10] R. E. Collin, *Foundations for Microwave Engineering*, McGraw-Hill, 1966.

[11] R. E. Collin, *Field Theory of Guided Waves*, IEEE PRESS, 1991.

[12] A. N. Novokhatski and A. Mosnier, "Short bunch wake potentials for a chain of TESLA cavities", DAPNIA/SEA-96-08, 1996.

# Appendix

In order to discuss the inverse Fourier transformation of $Z'(k_0)$ according to Eq. (10) it is convenient to consider an equivalent problem. Let $u^\delta(t)$ denote the inverse Fourier transformation of the input impedance $Z(\omega)$ corresponding to the circuit shown in Fig. 17:

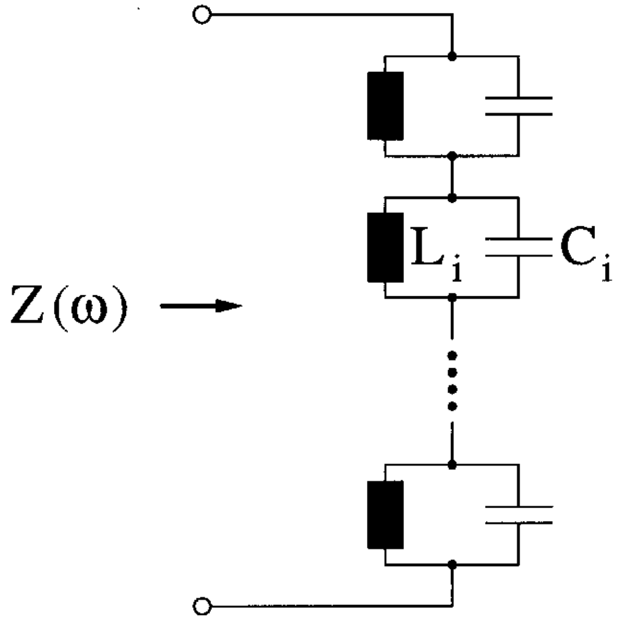


Figure 17: LC parallel networks which are connected in series.

$$u^\delta(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} Z(\omega)\, e^{j\omega t} d\omega \tag{18}$$

$u^\delta(t)$ is just the voltage response of the network due to a $\delta$-current excitation.

The input impedance $Z(\omega)$ of the network shown in Fig. 17 is given by

$$Z(\omega) = j \sum_{i=1}^{\infty} \frac{\frac{\omega}{\omega_i}}{1 - \left(\frac{\omega}{\omega_i}\right)^2} Z_i \quad \text{with} \quad Z_i = \sqrt{\frac{L_i}{C_i}} \quad \text{and} \quad \omega_i = \frac{1}{\sqrt{L_i C_i}} \quad . \tag{19}$$

If we compare the above equation with Eq. (8) we see that that $Z(\omega)$ and $Z'(k_0)$ are equivalent.

Note that the poles of $Z(\omega)$ are located at $\pm\omega_i$ exactly on the $\omega$-axis. It is therefore not straightforward to find $u^\delta(t)$ by the inverse Fouriertransformation according to Eq. (18). For the evaluation of the Fourier integral one would have to choose the path of integration around the poles such that the voltage response is causal as it has to be for a real network.

On the other hand, $u^\delta(t)$ can alternatively be calculated in time domain. $u^\delta(t)$ is the sum of all $u_i^\delta(t)$ which are the voltage responses of the individual LC parallel networks:

$$u^\delta(t) = \sum_{i=1}^{\infty} u_i^\delta(t) \tag{20}$$

Each $u_i^\delta(t)$ satisfies the second order differential equations

$$\left(\frac{d^2}{dt^2}+\omega_i^2\right)u_i^\delta(t)=0 \tag{21}$$

under the initial value conditions

$$u_i^\delta(t)\Big|_{t=0+0} = \frac{1}{C_i} \quad , \tag{22}$$

$$\frac{d}{dt}\,u_i^\delta(t)\Big|_{t=0+0} = 0 \quad , \tag{23}$$

where $t=0+0$ denotes $t$ just after the $\delta$-current excitation. Note that the charge of the current pulse has been assumed to be 1. The well-known solution for $u_i^\delta(t)$ reads

$$u_i^\delta(t)=\omega_i Z_i \cos(\omega_i t) \quad . \tag{24}$$

Consequently, we get for $u^\delta(t)$

$$u^\delta(t)=\sum_{i=1}^{\infty}\omega_i Z_i \cos(\omega_i t) \quad . \tag{25}$$

If we insert the above equation and Eq. (19) into Eq. (18), we finally obtain

$$\sum_{i=1}^{\infty}\omega_i Z_i \cos(\omega_i t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(j\sum_{i=1}^{\infty}\frac{\dfrac{\omega}{\omega_i}}{1-\left(\dfrac{\omega}{\omega_i}\right)^2}Z_i\right)e^{j\omega t}d\omega \quad . \tag{26}$$

In the previous formula the poles of $Z(\omega)$ are automatically taken into account in such a way that the voltage response is causal since Eq. (26) has been derived for a physical network. This is exactly what is required for the inverse Fourier transformation of $Z'(k_0)$ according to Eq. (10). $W^{\delta\prime}(s)$ which is given in Eq. (11) follows then directly from Eq. (26).