Bunchlengthening in Tesla Damping Ring

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In this short note we discuss about the bunchlengthening in the Tesla Damping ring by an analytic method. The method first solves the Haissinski equation and obtains the single-particle action-angle variables (J, ϕ) under the rf acceleration and wake potential. Then, it solves the eigenvalue of the linearized Vlasov equation fully including the potential well distortion. The distribution $\psi(J, \phi)$ is expanded into a Fourier series $\sum e^{im\phi}$ and the J-dependence is expressed by a mesh (equally-spaced in the amplitude $\sqrt{2J}$). Non-zero imaginary part of the eigenvalue indicates instability (growth of the energy spread). The detail of the employed algorithm is decribed in the Appendix.

The relevant ring parameters are summarized in Table.1, which is based on the old design with circumference 20km. The beam energy is 3GeV and the design current is 0.09mA in all cases.

Table 1: Ring Parameters

	A	B	C	
Momentum compaction parameter α	27×10^{-5}	7×10^{-5}	5×10^{-5}	
Natural energy spread σ_{E0}	3.44	3.44	3.44	MeV
Synchrotron oscillation frequency f_s	1.304	0.6752	0.5706	kHz
Natural bunchlength σ_{z0}	10.1	5.161	4.362	mm

The wake Green function, which has been prepared by C. Burnton, is plotted in Figure.1. It is the sum of the wakes in bellows, rf cavities, kickers, and resistive wall by a Gaussian bunch of r.m.s. length 0.5mm (2mm length was used for the kicker wake). The wake is non-zero for $s < 0$ because of the finite length of the bunch. (The algorithm accepts 'acausal' part of wake.) As one can see the Green function is dominated by an inductive component, which mainly comes from the bellow, plus a small resistive component.

Figure.2 shows the center-of-mass position of the equilibrium bunch of Haissinski solution with respect to the unperturbed synchronous position as a function of the bunch current. The negative value means that the bunch goes forward. Figure.3 shows the equilibrium r.m.s. bunch length. In both figures the crosses on the curves are the point where the system becomes unstable (threshold).

Figure.4 is the growth rate in 1/sec.

In the short bunch cases the convergence of the threshold current and the growth rate with respect to the truncation of the Fourier harmonics m_{max} is not good. ($m_{max} = 4, 6, 10$ is used for the cases A,B,C, respectively.) In the case C the instability comes from very high harmonics ($m = 5$ or 6). The threshold in such cases depends on the detail of the s-dependence of the Green function near the origin.

As one can see the threshold currents in all cases are much higher than the design current at least by a factor 5, albeit the theory still has some uncertainty. In the case of JLC/NLC the threshold is only factor of 2 above the design current in spite that the bunch charge is 3 times lower. The large difference seems to come from the fact that the major portion of the TESLA damping ring is vaccuum chamber with large crosssection so that the average impedance per unit length is considerably lower.

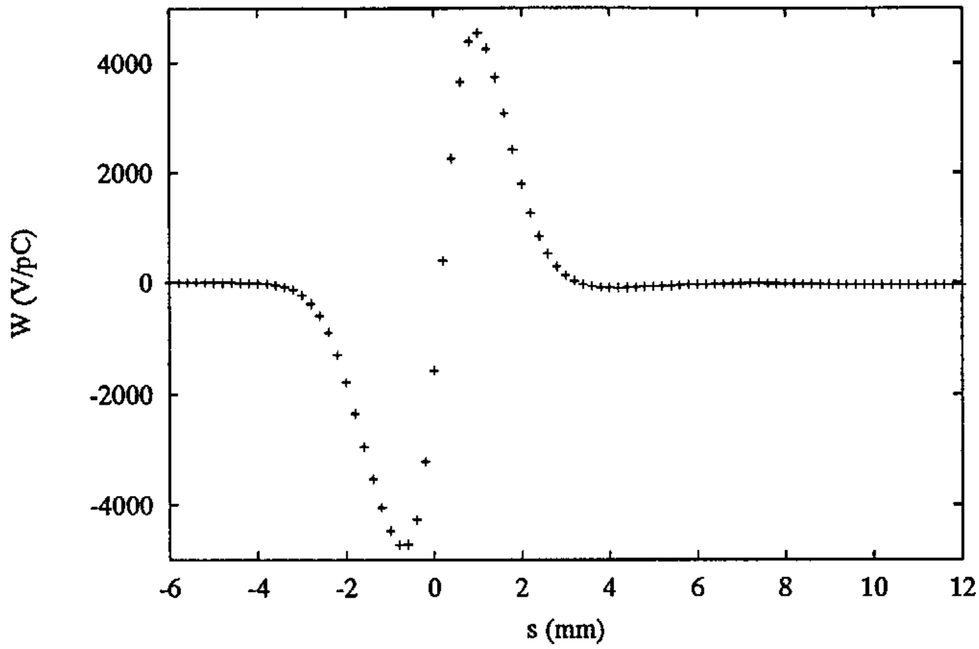

Figure 1: The wake Green function used in the calculation.

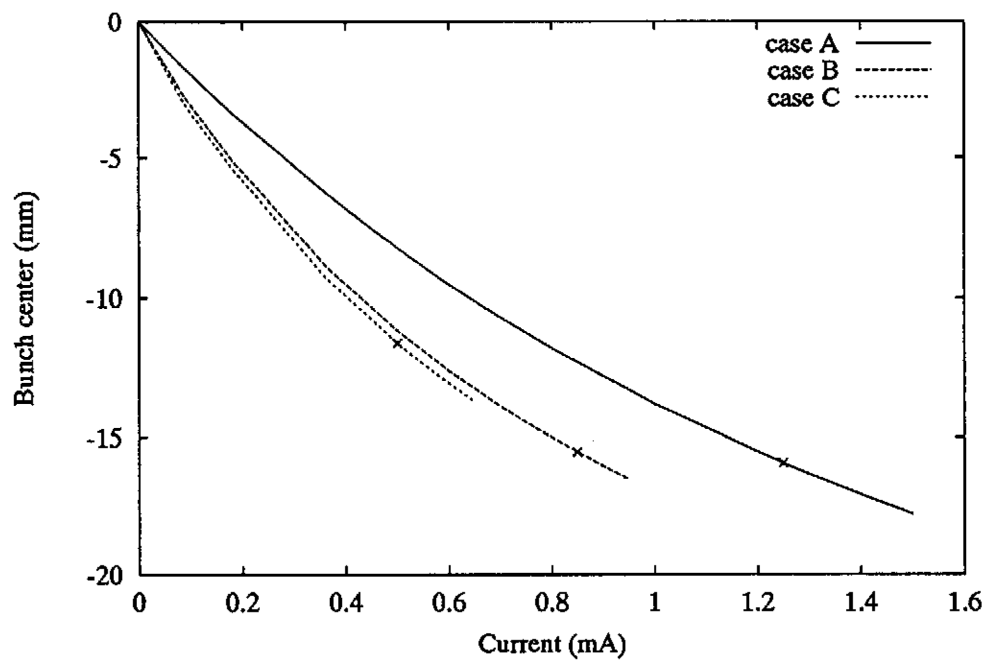

Figure 2: The bunch center-of-mass position in equilibrium with respect to the natural synchronous position. The negative value means that the bunch moves to the head. The crosses indicate the threshold of energy-spread increase.

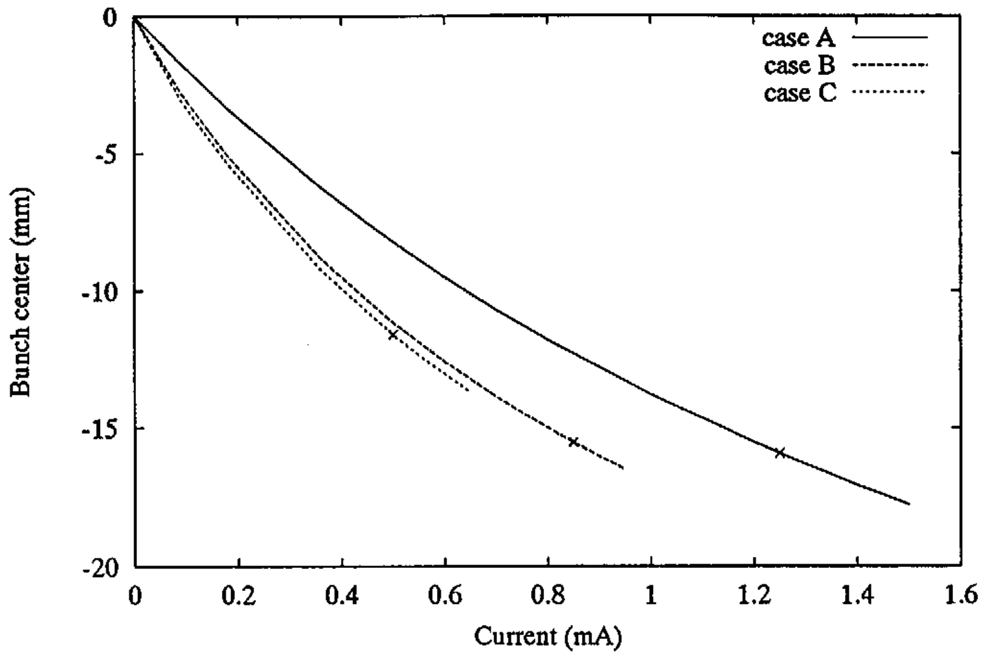

Figure 3: The r.m.s bunch length from Haissinski equation. The crosses indicate the threshold of energy-spread increase.

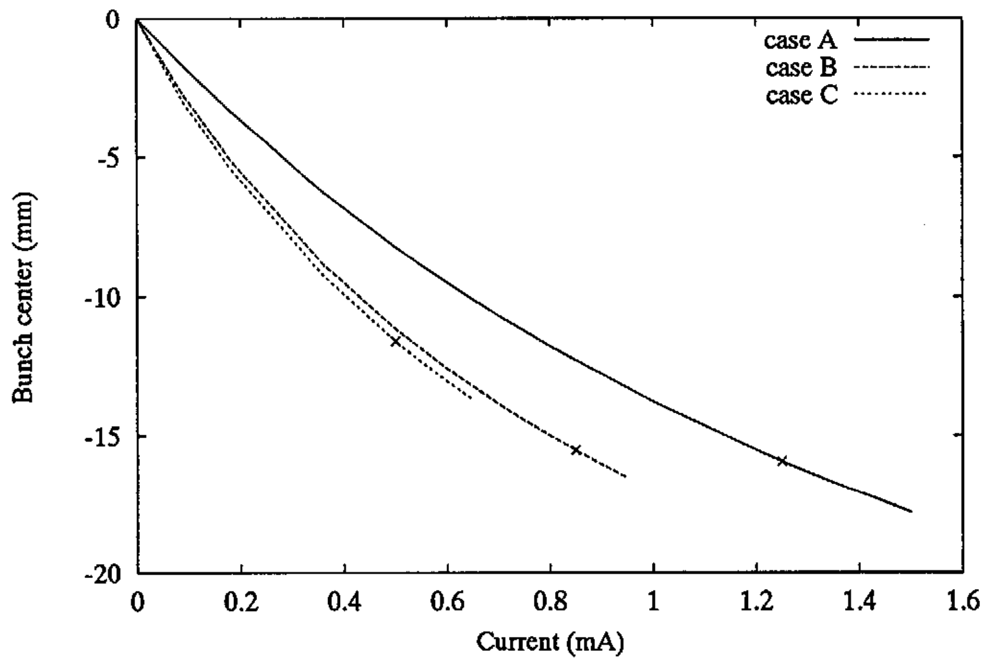

Figure 4: The growth rate in units of 1/sec as a function of current.

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A Appendix

The formalism used in this note was developed in 1990 by K. Oide and myself but was not published. It is described in [2] but is not transparent because the purpose of the paper is different. I shall briefly describe the formalism here.

We define p as the off-energy in units of the natural energy spread σ_{E0} and q as the longitudinal position from the synchronous position in units of the natural bunchlength σ_{z0} (positive behind the synchronous position). We choose $\theta = \omega_{s0}t$, the phase angle of natural synchrotron oscillation, as the independent variable.

The equation of a single particle under the influence of longitudinal wake is written as

$$\begin{aligned} \frac{dq}{d\theta} &= p \\ \frac{dp}{d\theta} &= -q - \int_{-\infty}^{q} W(q-q')\Psi(p',q')dp'dq'. \end{aligned} \tag{1}$$

Here Ψ is the distribution function (normalized as $\int \Psi(p,q)dpdq = 1$) and $W(q)$ is the normalized wake Green function

$$W(q) = \frac{I}{\omega_{s0}\sigma_{E0}} w(\sigma_{z0}q) \tag{2}$$

where I is the average current and w is the Green function (Volt/Coulomb/turn) (positive for deceleration). We ignore the sinusoidal curve of the rf voltage and possible non-uniform distribution of the wake around the ring.

From eq.(1) one gets the Vlasov-Fokker-Planck equation

$$\frac{\partial \Psi}{\partial \theta} = -p\frac{\partial \Psi}{\partial q} + \frac{\partial \Psi}{\partial p}\left[q + \iint_{-\infty}^{\infty} dp'dq'W(q-q')\Psi(p',q',\theta)\right] + \mathcal{R}\Psi \tag{3}$$

Here we included the radiation effect

$$\mathcal{R}\Psi = 2\lambda_D \frac{\partial}{\partial p}\left(\frac{\partial \Psi}{\partial p} + p\Psi\right), \qquad \lambda_D = \frac{1}{\tau_D \omega_{s0}} \tag{4}$$

where τ_D is the damping time.

As usual, eq.(3) is solved in two steps. First find a static solution $\Psi_0(p,q)$ and study the stability of the linear perturbation from Ψ_0. The equation for Ψ_0 is reduced to the well-known Haissinski equation

$$\Psi_0(p,q) = \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p}\rho_0(q), \qquad \frac{d\rho_0}{dq} = -\frac{\rho_0(q)}{\sigma_p^2}\left[q + \int_{-\infty}^{\infty} dq'W(q-q')\rho_0(q')\right] \tag{5}$$

with $\sigma_p = 1$ and the linearized Vlasov equation for $\psi(p,q,\theta) = \Psi - \Psi_0$ is

$$\begin{aligned} \frac{\partial \psi}{\partial \theta} &= -p\frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial p}\left[q + \iint_{-\infty}^{+\infty} dp'dq'W(q-q')\Psi_0(p',q')\right] \\ &\quad + \frac{\partial \Psi_0}{\partial p}\iint_{-\infty}^{+\infty} dp'dq'W(q-q')\psi(p',q',\theta) + \mathcal{R}\psi. \end{aligned} \tag{6}$$

The key of our method is to solve this equation including the potential-well distortion term $W \times \Psi_0$, which is usually ignored. To do so, we introduce action-angle variable.

For the static line density $\rho_0(q)$, the single-particle equation of motion (1) can be derived from the single-particle Hamiltonian

$$H(p,q) = \frac{1}{2}p^2 + V(q), \qquad V(q) = \frac{1}{2}q^2 + \int_{-\infty}^{\infty} U(q-q')\rho_0(q')dq' \tag{7}$$

where

$$U(q) = \int_{-\infty}^{q} W(q)dq. \tag{8}$$

We can define the action-angle variables by

$$J = \frac{1}{2\pi}\oint \sqrt{2[H - V(q)]}dq, \tag{9}$$

$$\phi = \omega(J)\int_{q_{min}(J)}^{q} \frac{dq}{\sqrt{2[H-V(q)]}}. \tag{10}$$

Here, $q_{min}(J)$ is the minimum q on the trajectory of given J. (This means $q = -\sqrt{2J}\cos\phi$ and $p = \sqrt{2J}\sin\phi$ in the absense of the wake.)

Then, the Hamiltonian is a function of J only:

$$H(J) = \int_0^J \omega(J)dJ \tag{11}$$

where $\omega(J)$ is the single-particle angular frequency. The canonical transformation (p,q) to (J,ϕ) can be obtained from the generating function

$$F(J,q) = \int^q dq\sqrt{2\big(H(J) - V(q)\big)}. \tag{12}$$

Note that the relations

$$\frac{\partial\phi}{\partial q} = \frac{\omega(J)}{p}, \qquad \frac{\partial J}{\partial p} = \frac{p}{\omega(J)} \tag{13}$$

hold exactly.

By using (J,ϕ) instead of (p,q), one can rewrite the linearized Vlasov equation as

$$\frac{\partial\psi}{\partial t} = -\omega(J)\frac{\partial\psi}{\partial\phi} + \frac{\partial\Psi_0}{\partial p}\int W(q-q')\psi(p',q',t)dp'dq' + \mathcal{R}\psi. \tag{14}$$

Let us rewrite eq.(14) in a more convenient form. First, note that, for any function f of q only, one finds

$$\frac{df(q)}{dq} = \left.\frac{\partial\phi}{\partial q}\right|_J \left.\frac{\partial}{\partial\phi}\right|_J f(q(J,\phi)) = \frac{\omega(J)}{p}\left.\frac{\partial}{\partial\phi}\right|_J f(q(J,\phi)) \tag{15}$$

since

$$\left.\frac{\partial \phi}{\partial q}\right|_J = \frac{\omega(J)}{p}. \tag{16}$$

Using this, one can rewrite the integral in Eq.(14) as

$$\begin{aligned}\frac{\partial \Psi_0}{\partial p}\int W(q-q') \quad & \psi(p',q',\theta)dp'dq' \\ &= \frac{\partial H}{\partial p}\frac{\partial J}{\partial H}\frac{\partial \Psi_0}{\partial J}\frac{d}{dq}\int U(q-q')\psi(p',q',\theta)dp'dq' \\ &= \frac{\partial \Psi_0}{\partial J}\left.\frac{\partial}{\partial \phi}\right|_J\int U(q(J,\phi)-q(J',\phi'))\psi(\bar{J}',\bar{\phi}',\theta)dJ'd\phi'.\end{aligned}$$

Thus, Eq.(14) is now

$$\frac{\partial \psi}{\partial \theta} = -\omega(J)\frac{\partial \psi}{\partial \phi} + \frac{\partial \Psi_0}{\partial J}\left.\frac{\partial}{\partial \phi}\right|_J\int U(q(J,\phi)-q(J',\phi'))\psi(J',\phi',\theta)dJ'd\phi' + \mathcal{R}\psi. \tag{17}$$

In solving this equation we expand ψ into a fourier series of the angle variable:

$$\psi(J,\phi,\theta) = \sum_{m=-\infty}^{+\infty} e^{im\phi-i\Omega\theta}\psi_m(J) \tag{18}$$

Then,

$$-i\Omega\psi_m(J) = -im\omega(J)\psi_m + im\frac{\partial \Psi_0}{\partial J}\sum_{m'}{}'\int K_{m,m'}(J,J')\psi_{m'}(J')dJ'. \tag{19}$$

The kernel $K_{m,m'}$ is defined as

$$K_{m,m'}(J,J') = \frac{1}{2\pi}\oint d\phi e^{-im\phi}\oint d\phi' e^{im'\phi'}U\big(q(J,\phi)-q(J',\phi')\big). \tag{20}$$

Owing to the choice of the origin of ϕ, $q(J,\phi)$ is an even function of ϕ. Therefore, $K_{m,m'}$ is real and satisfies

$$K_{m,m'} = K_{-m,m'} = K_{m,-m'} = K_{-m,-m'}. \tag{21}$$

It has no general symmetry with respect to the exchange of J and J'. When the wake Green function $W(q)$ is given in the form of the impedance

$$W(q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Z(k)e^{-ikq}dk, \qquad Z(k) = \int_{-\infty}^{+\infty} W(q)e^{ikq}dq, \tag{22}$$

the Fourier transform of $U(q)$ is written as

$$U(q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{U}(k)e^{-ikq}dk, \qquad \hat{U}(k) = \int_{-\infty}^{+\infty} U(q)e^{ikq}dq = \frac{iZ(k)}{k+i0}. \tag{23}$$

Then, the kernel $K_{m,m'}$ can be expressed as

$$K_{m,m'}(J,J') = \int_{-\infty}^{+\infty} dk\hat{U}(k)L_m^*(k,J)L_{m'}(k,J'), \tag{24}$$

where

$$L_m(k,J) = \frac{1}{2\pi}\oint d\phi e^{im\phi+ikq(J,\phi)}. \tag{25}$$

At small J, $L_m(k,J)$ behaves as $J^{(|m|/2)}$ so does $K_{m,m'}$ as $J^{(|m|/2)}J'^{(|m'|/2)}$.

Eq.(19) contains both positive and negative m but can be reduced owing to the symmetry (21). First, define

$$f_m(J) = \psi_m + \psi_{-m}. \tag{26}$$

Then eq.(19) splits into two equations

$$\Omega f_m(J) - m\omega(J)[\psi_m - \psi_{-m}] = 0 \tag{27}$$

$$\Omega[\psi_m - \psi_{-m}] - m\omega(J)f_m(J) = -2m\Psi_0' \sum_{m'=1}^{\infty}\int K_{m,m'}(J,J')f_m dJ', \tag{28}$$

(we have omitted here the radiation term but it is included in actual calculation.) Eqs.(27) and (26) give

$$\psi_m(J) = \frac{m\omega(J)+\Omega}{2m\omega(J)}f_m(J). \qquad (m >< 0). \tag{29}$$

and Eq.(28) gives our final equation

$$\Omega^2 f_m(J) = m^2\omega(J)^2 f_m(J) - 2m^2\omega(J)\Psi_0'(J)\sum_{m'=1}^{\infty}\int K_{m,m'}(J,J')f_m(J')dJ'. \tag{30}$$

In order to reduce this integral equation to a finite matrix eigenvalue problem we introduce a mesh of J, J_j ($J = 1, 2 \ldots N_j$) so that integral over J is repalced by a sum over J_j (We choose equal-space mesh with respect to $\sqrt{J}$).

As usual, the system is considered to be stable if all the eigenvalues are real. Otherwise, we need another step to find equilibrium bunchlength, which is the subject of [2]. However, a conclusion of [2] is that, unless too high above the threshold current, the following method is a good approximation. Solve eq.(5) by increasing $\sigma_p > 1$ until all the eigenvalues become real. Then you get a new equilibrium.

References

[1] Oide, K. and Yokoya, K., KEK-preprint-90-10 (1990).

[2] The Tamura Symposium Proceedings, *The future of Accelerator Physics* Austin, Texas, Nov. 1994, AIP conference proceedings 356, page.127