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On the upper limit of the rms energy width due to beamstrahlung and its numerical simulation

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Consider a particle energy distribution $f(x)$ after beam-beam interaction as shown in Fig. 1(a). x denotes the fractional energy loss $\delta E/E_0$ due to beamstrahlung. Let $\overline{x} = \langle x \rangle$ denote the average fractional energy loss and σ_x^2 its variance. In numerical simulations, it is sometimes observed that σ_x is larger than $\overline{x}$, and it might be interesting, whether there is a principal upper limit on σ_x at given $\overline{x}$. If $F(x)$ and $\varphi(x)$ denotes the first and second integral of $f(x)$, respectively, and if $f(x)$ is normalized to unity, it is seen by partial integration, that

$$\overline{x} = \int_0^\infty x f(x) dx = xF|_0^\infty - \int_0^\infty F dx = (x - \varphi)|_0^\infty \tag{1}$$

A simple geometric interpretation of of the quantity $\overline{x}^2/2$ is given by the dotted area in Fig.1c.

Similarly, it is seen that

$$\sigma_x^2 = \langle x^2 \rangle - \overline{x}^2 = 2 \int_0^\infty \varphi dx - \varphi^2|_0^\infty \tag{2}$$

Therefore, the geometric interpretation of $\sigma_x^2/2$ is just the hatched area in Fig.1c.

Because of $F(x \to \infty) = 1$, the asymtote of φ must have a slope equal to unity. Since f is positive everywhere, F is monotonous, and φ is <u>above</u> its asymptote everywhere.

Now consider the ratio $R = (\sigma_x^2/2)/(\overline{x}^2/2) = \sigma_x^2/\overline{x}^2$. From its geometric interpretation (Fig.1c) it is obvious that R could be infinitely large if the maximum value of x with nonzero probability (denoted by $\hat{x}$ in the following) is infinite. However, considering beamstrahlung, $\hat{x}$ is limited. Then, the maximum value of $\sigma_x^2/2$ is given by the triangle $A, \overline{X}$ in Fig.1c. Thus,

$$R_{max} = \frac{(\hat{x} - \overline{x}) \cdot x/2}{\overline{x}^2/2} = \frac{\hat{x}}{\overline{x}} - 1 \tag{3}$$

or

$$\frac{\sigma_x}{\overline{x}} < \sqrt{\frac{\hat{x}}{\overline{x}} - 1} \tag{4}$$

For the principal upper limit $\hat{x} = 1$ and a typical $\overline{x} = 5\%$ we get $\frac{\sigma_x}{\overline{x}} < 4.4$. For the ratio to attain that maximum value, however, the distribution function $f(x)$ would have to look very funny (see Fig.2). Above all, the distribution function $f(x)$ would have to be non-monotonous (see Fig.2a). For any physically reasonable distribution function, $\sigma_x/\overline{x}$ would have to be much smaller. It is worth noting that a distribution function as illustrated in Fig.2a is characteristic for a numerical simulation with a too small number N_0 of particles. The effect of just one missing (macro-) particle between $\overline{x}$ and $\hat{x}$ is illustrated in Fig.2c. Because the slope of φ is constant where f is zero (no particle), any additional (macro-) particle between $\overline{x}$ and $\hat{x}$ would considerably reduce R. A very rough estimate of the fluctuation of R due to such a fluctuation of the particle distribution can also be found from Fig.2c: it is the hatched area, divided by $\overline{x}^2/2$. The hatched area is, roughly speaking, given by $\delta\varphi' \cdot \overline{x}^2/2$, with $\delta\varphi'$ being the change in slope of φ at $\hat{x}$, if one particle is added. Since $\delta\varphi' = \delta F = \delta N/N_0 = 1/N_0$, we get

$$\delta R \approx \frac{\delta\varphi' \cdot \hat{x}^2/2}{\overline{x}^2/2} \approx \frac{\hat{x}^2}{N_0\overline{x}^2} \tag{5}$$

Using $\hat{x} = 1$, we get

$$\delta R \approx \frac{1}{N_0\overline{x}^2} \tag{6}$$

If we want to reduce the statistical fluctuation of R from seed to seed well below unity, we get

$$\delta R \approx \frac{1}{N_0\overline{x}^2} \ll 1 \tag{7}$$

or

$$N \gg (\overline{x})^{-2} \tag{8}$$

For example, if the average fractional energy loss is 2%, one has to track much more than 2500 particles to be sure that $\sigma_x/\overline{x}$ would not fluctuate by more than unity from seed to seed.

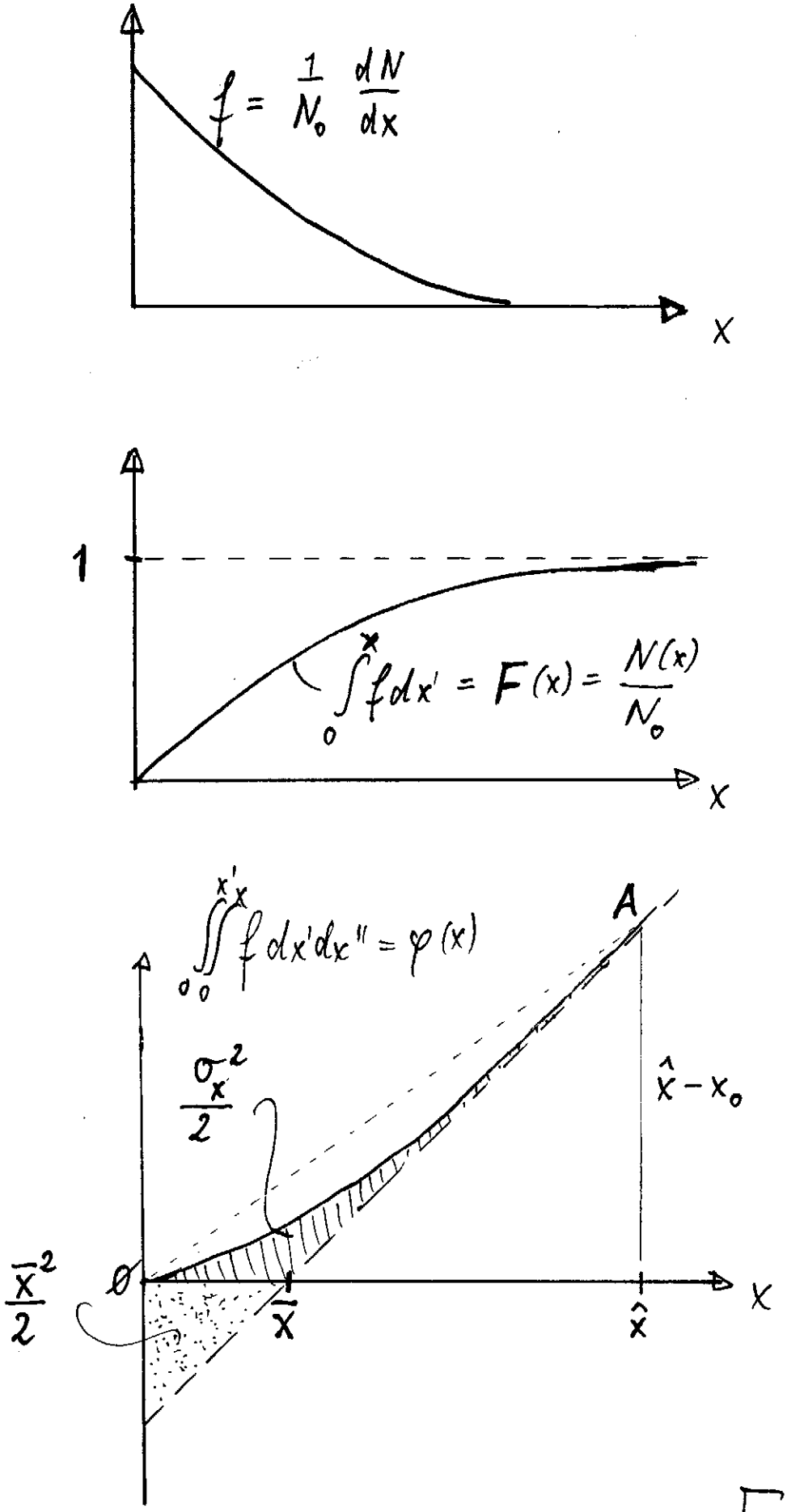

Fig. 1, see text

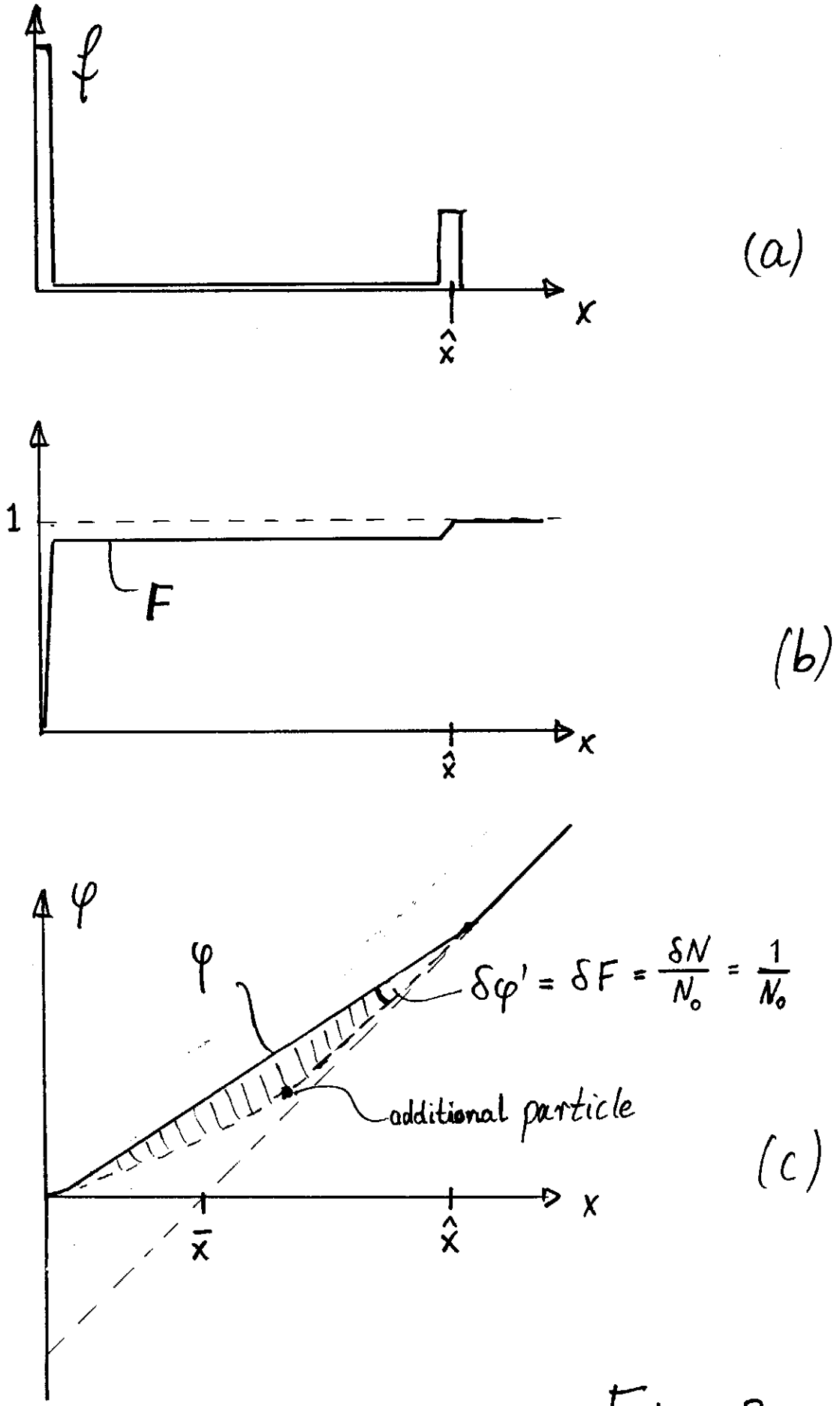

Fig. 2, see text