Envelope Equations for Transients in Linear Chains of Resonators

H. Henke and M. Filtz
Technische Universität, EN-2, Einsteinufer 17, D-1000 Berlin 10, Germany

June 3, 1993

Abstract

Transient signals in strings of resonators consist of regimes with different time constants: high frequency oscillations, beat signals and exponentials. If one is interested only in the signals envelope one can transform the system of second order differential equations into a system of first order differential equations. The later carries fast varying terms, which are averaged out, and slowly varying terms. The resulting equations are well behaving and can be integrated numerically. Results are shown for the filling process under beam loading of the superconducting nine–cell TESLA cavity.

1 Introduction

Transients in strings of resonators are usually calculated by means of a Laplace transform in matrix notation or by a discrete Laplace transform (see for instance [1]). Both approaches become quite awkward if the string is not homogenous and/or has branches. Also, one is often not interested in the full time response but only the signals envelope. Then, it may be convenient to take advantage of the fact that the system consists of three regimes with normally very different time constants: First, the high frequency oscillations with the time constant T_{RF} of one period. Second, beating signals with time constants T_{RF}/k where k is the coupling between resonators. Third, signals which are related to the filling time Q/ω_{RF}.

In the following it is shown how to transform the system of second order differential equations (DE) describing the individual resonators into a system of first order DE's of twice the size. The system is written in a way that the fast varying terms can be averaged out and only slowly varying terms remain. The left over system of DE's is integrated numerically yielding the signal envelopes.

The method is applied to the filling process of the superconducting TESLA cavity consisting of nine resonators. Due to the high Q of the cavity the filling time is of the order of one ms whereas the RF period is less than one ns. The coupling between cells is in the percent region. Thus, the time constants are well separated and the proposed method is ideally suited.

2 Filling of a Single Resonator

Let us assume a single resonator which is driven by a generator via some coupling device, Fig. 1.

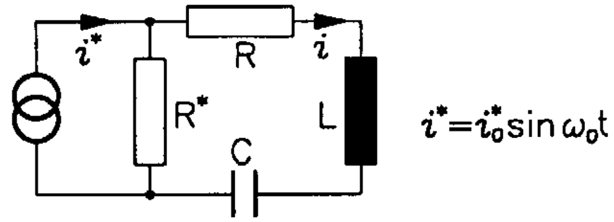

Figure 1: Single resonator driven by a generator with transformed current i^* and internal impedance R^*.

i^* and R^* are the generators current and impedance transformed by the coupling device. The loop equation of the circuit can be written as

$$\ddot{q} + \frac{\omega_0}{Q_L}\dot{q} + \omega_0^2 q = f_0 \sin\omega_0 t \tag{1}$$

$$q = \int i\,\mathrm{d}t\ ,\ \omega_0 = \frac{1}{\sqrt{LC}}\ ,\ Q_L = \frac{\omega_0 L}{R + R^*}\ ,\ f_0 = \frac{R^*}{L} i_0^*\ .$$

For q we try an ansatz called variation of constants

$$q(t) = a(t)\ \cos\omega_0 t + b(t)\ \sin\omega_0 t \quad . \tag{2}$$

(1), (2) are two equations for three unknown functions q, a, b. Hence, we can impose a third condition which we choose as

$$\dot{a}\ \cos\omega_0 t + \dot{b}\ \sin\omega_0 t = 0\ . \tag{3}$$

Differentiation of (2) while considering (3) and substituting into (1) gives

$$(\dot{b} + \frac{\omega_0 b}{Q_L})\cos\omega_0 t - (\dot{a} + \frac{\omega_0 a}{Q_L})\sin\omega_0 t = \frac{f_0}{\omega_0}\sin\omega_0 t\ . \tag{4}$$

Now, multiplying (4) with $\sin\omega_0 t$ and (3) with $\cos\omega_0 t$ we can eliminate $\dot{b}$ through substraction. In a similar way we eliminate $\dot{a}$ and obtain a system of first order DE's

$$\dot{a} + \frac{\omega_0}{2Q_L}a + \frac{f_0}{2\omega_0} = \frac{\omega_0}{2Q_L}(a\ \cos 2\omega_0 t + b\ \sin 2\omega_0 t) + \frac{f_0}{2\omega_0}\cos 2\omega_0 t$$

$$\dot{b} + \frac{\omega_0}{2Q_L}b = \frac{\omega_0}{2Q_L}(a\ \sin 2\omega_0 t - b\ \cos 2\omega_0 t) + \frac{f_0}{2\omega_0}\sin 2\omega_0 t\ . \tag{5}$$

So far, equ. (5) is still exact. We only have transformed the second order DE (1) for q into two first order DE's for a and b. Not much seems to be gained. But (5) is well suited to determine approximately a and b if they are slowly varying, i.e. if they do not change much over one period $T_0 = 2\pi/\omega_0$. Then, we can average the equs. (5) over T_0 and the right sides become zero. The solutions of the remaining left sides are straight forward

$$a = \frac{Q_L}{\omega_0^2} f_0 \left(\mathrm{e}^{-t/\tau} - 1\right) \ , \ \tau = 2Q_L/\omega_0 \ , \ b = 0 \tag{6}$$

where we have used the initial conditions $q(t=0) = \dot{q}(t=0) = 0$.

For a single resonator we could have derived the envelope equations (6) easily in different ways. So, it is hardly worth mentioning it, if it were not to explain the procedure for the more complicated cases of strings of resonators.

3 Transients in a Chain of Coupled Resonators

Next, we consider a chain of N coupled resonators, Fig 2.

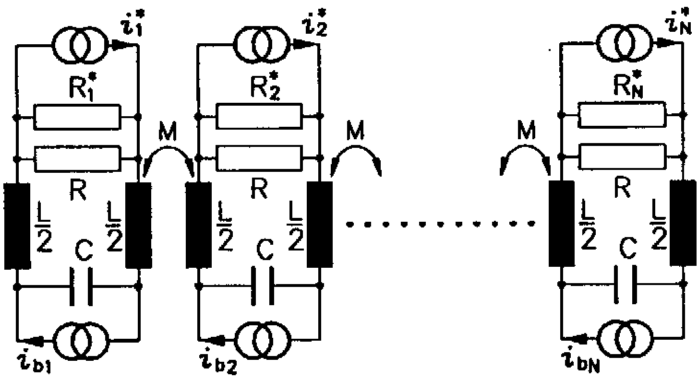

Figure 2: Chain of N coupled resonators driven by generator currents i_n^* and beam currents i_{bn}.

Each resonator is coupled to a generator with transformed current i_n^* and impedance R_n^*. The beam currents i_{bn} are assumed to be δ–function like, so they can be taken into account as a jump in q_n at any instant t. Then the second order DE's for each loop can be written in a matrix notation

$$\ddot{\mathbf{q}} + \frac{\omega_0}{Q_0}(\mathbf{I} + \boldsymbol{\beta})\dot{\mathbf{q}} + \omega_0^2\mathbf{q} - k\mathbf{K}\ddot{\mathbf{q}} = \mathbf{f} \tag{7}$$

with

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} , \ \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 & \cdots \\ 0 & \beta_2 & \\ \vdots & & \ddots \end{bmatrix}$$

$$\beta_n = R_n^*/R \ , \ \omega_0^2 = 1/LC \ , \ Q_0 = \omega_0 L/R \ , \ k = L/M$$

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ \vdots & & & & \ddots \end{bmatrix} , \ \mathbf{f} = \begin{bmatrix} f_1 \sin\omega_{01}t \\ f_2 \sin\omega_{02}t \\ \vdots \end{bmatrix} .$$

In order to solve the system (7) we try an ansatz

$$\mathbf{q} = \mathbf{Q}[\mathbf{C}\mathbf{a} + \mathbf{S}\mathbf{b}]\,,\ \dot{\mathbf{q}} = \mathbf{Q}[\mathbf{C}\dot{\mathbf{a}} + \mathbf{S}\dot{\mathbf{b}} - \boldsymbol{\omega}\mathbf{S}\mathbf{a} + \boldsymbol{\omega}\mathbf{C}\mathbf{b}] \tag{8}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \end{bmatrix}\ \mathbf{Q} = \begin{bmatrix} q_1^{(1)} & q_1^{(2)} & \cdots \\ q_2^{(1)} & q_2^{(2)} & \\ \vdots & & \ddots \end{bmatrix}$$

and $\boldsymbol{\omega}$, $\mathbf{C}$, $\mathbf{S}$ are diagonal matrices with elements ω_i, $\cos\omega_i t$, $\sin\omega_i t$, respectively. $a_i(t)$, $b_i(t)$ are the slowly varying amplitudes in each cell and ω_i, $\mathbf{q}^{(i)}$ are the eigenfrequencies and eigenvectors of the steady–state, loss–free, homogenous system respectively. The latter are derived in the appendix. Similar to the case of a single resonator, we impose the condition

$$\mathbf{C}\dot{\mathbf{a}} + \mathbf{S}\dot{\mathbf{b}} = 0 \tag{9}$$

in order to reduce the degree of freedom for the functions in (8). After differentiating (8) and making use of (9) we substitute into (7) and find

$$\mathbf{A}(\mathbf{C}\dot{\mathbf{b}} - \mathbf{S}\dot{\mathbf{a}}) + \frac{\omega_0}{Q_0}(\mathbf{I} + \boldsymbol{\beta})\mathbf{Q}\boldsymbol{\omega}(\mathbf{C}\mathbf{b} - \mathbf{S}\mathbf{a}) + \mathbf{M}(\mathbf{C}\mathbf{a} + \mathbf{S}\mathbf{b}) = \mathbf{f} \tag{10}$$

with $\mathbf{A} = (\mathbf{I} - k\mathbf{K})\mathbf{Q}\boldsymbol{\omega}$, $\mathbf{M} = \omega_0^2\mathbf{Q} - \mathbf{A}\boldsymbol{\omega} = 0$.
$\mathbf{M}$ is the system matrix of the steady–state, loss–free, homogenous case and thus vanishes.

Because of the unitarian character of $\mathbf{Q}$, $\mathbf{Q}^{-1} = \mathbf{Q}^t$, we can invert $\mathbf{A}$ and obtain for (10)

$$\mathbf{C}\dot{\mathbf{b}} - \mathbf{S}\dot{\mathbf{a}} + \frac{\boldsymbol{\omega}}{\omega_0 Q_0}(\boldsymbol{\omega} + \mathbf{P}\boldsymbol{\omega})(\mathbf{C}\mathbf{b} - \mathbf{S}\mathbf{a}) = \frac{1}{\omega_0^2}\boldsymbol{\omega}\mathbf{Q}^t\mathbf{f} \tag{11}$$

with $\mathbf{P} = \mathbf{Q}^t\boldsymbol{\beta}\mathbf{Q}$. Successive elimination of $\dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ from (9) and (11) yields

$$\begin{aligned} \dot{\mathbf{a}} + \frac{1}{\omega_0 Q_0}\left[\boldsymbol{\omega}^2(\mathbf{S}^2\mathbf{a} - \mathbf{S}\mathbf{C}\mathbf{b}) + \boldsymbol{\omega}\mathbf{S}\mathbf{P}(\mathbf{S}\boldsymbol{\omega}\mathbf{a} - \mathbf{C}\boldsymbol{\omega}\mathbf{b})\right] &= -\frac{1}{\omega_0^2}\boldsymbol{\omega}\mathbf{S}\mathbf{Q}^t\mathbf{f} \\ \dot{\mathbf{b}} + \frac{1}{\omega_0 Q_0}\left[\boldsymbol{\omega}^2(\mathbf{C}^2\mathbf{b} - \mathbf{C}\mathbf{S}\mathbf{a}) + \boldsymbol{\omega}\mathbf{C}\mathbf{P}(\mathbf{C}\boldsymbol{\omega}\mathbf{b} - \mathbf{S}\boldsymbol{\omega}\mathbf{a})\right] &= \frac{1}{\omega_0^2}\boldsymbol{\omega}\mathbf{C}\mathbf{Q}^t\mathbf{f} \end{aligned} \tag{12}$$

In (12) we find again products of sine– and cosine–functions which we decompose into slowly and fast varying terms, e.g.

$$\sin(\omega_i t)\, cos(\omega_j t) = [\sin(\omega_i - \omega_j)t + \sin(\omega_i + \omega_j)t]/2\,.$$

Now, we average the system over a time span approximately equal to the period of the fast varying signals and obtain, finally, the system of first order DE's for the slowly varying signals

$$\begin{aligned} \dot{\mathbf{a}} + \frac{1}{2\omega_0 Q_0}[\boldsymbol{\omega}(\mathbf{I} + \mathbf{P}_c)\boldsymbol{\omega}\mathbf{a} - \boldsymbol{\omega}\mathbf{P}_s\boldsymbol{\omega}\mathbf{b}] &= -\frac{1}{2\omega_0^2}\boldsymbol{\omega}\mathbf{R}_c \\ \dot{\mathbf{b}} + \frac{1}{2\omega_0 Q_0}[\boldsymbol{\omega}(\mathbf{I} + \mathbf{P}_c)\boldsymbol{\omega}\mathbf{b} + \boldsymbol{\omega}\mathbf{P}_s\boldsymbol{\omega}\mathbf{a}] &= -\frac{1}{2\omega_0^2}\boldsymbol{\omega}\mathbf{R}_s \end{aligned} \tag{13}$$

where

$$\mathbf{P}_c = \begin{bmatrix} p_{11} & p_{12}\cos(\omega_1-\omega_2)t & p_{13}\cos(\omega_1-\omega_3)t & \cdots \\ p_{21}\cos(\omega_2-\omega_1)t & p_{22} & p_{23}\cos(\omega_2-\omega_3)t & \\ \vdots & & & \ddots \end{bmatrix}$$

$$\mathbf{P}_s = \begin{bmatrix} 0 & p_{12}\sin(\omega_1-\omega_2)t & p_{13}\sin(\omega_1-\omega_3)t & \cdots \\ p_{21}\sin(\omega_2-\omega_1)t & 0 & p_{23}\sin(\omega_2-\omega_3)t & \\ \vdots & & & \ddots \end{bmatrix}$$

$$\mathbf{R}_c = \begin{bmatrix} q_1^{(1)} f_1 \cos(\omega_1-\omega_{01})t + q_2^{(1)} f_2 \cos(\omega_1-\omega_{02})t + \ldots \\ q_1^{(2)} f_1 \cos(\omega_2-\omega_{01})t + q_2^{(2)} f_2 \cos(\omega_2-\omega_{02})t + \ldots \\ \vdots \end{bmatrix}$$

$$\mathbf{R}_s = \begin{bmatrix} q_1^{(1)} f_1 \sin(\omega_1-\omega_{01})t + q_2^{(1)} f_2 \sin(\omega_1-\omega_{02})t + \ldots \\ q_1^{(2)} f_1 \sin(\omega_2-\omega_{01})t + q_2^{(2)} f_2 \sin(\omega_2-\omega_{02})t + \ldots \\ \vdots \end{bmatrix}$$

p_{ij} are the elements of the matrix $\mathbf{P}$ defined in (11). The system (13) yields non–oscillating solutions and is very stable. It can easily be integrated over 10^5 periods $T = 2\pi/(\omega_i - \omega_j)$, for instance with a fifth order Runge–Kutta method. The initial conditions are normally given in $\mathbf{q}$ and $\dot{\mathbf{q}}$ and define $\mathbf{a}(0)$ and $\mathbf{b}(0)$ through (8) and (9).

The beam currents can be taken into acount by jumps in q_i and continuous $\dot{q}_i$, i.e. by $\delta\mathbf{q}(t_j)$ only. Then, from (8), (9) follows

$$\delta\mathbf{a}(t_j) = \mathbf{C}(t_j)\mathbf{Q}^t\delta\mathbf{q}(t_j)\,,\ \delta\mathbf{b}(t_j) = \mathbf{S}(t_j)\mathbf{Q}^t\delta\mathbf{q}(t_j)\,. \tag{14}$$

Having solved for $\mathbf{a}$, $\mathbf{b}$ we are still left to find reasonable envelopes from (8). Since, typically, the resonator chain is driven by a single generator with frequency ω_{0n} and the particles to be accelerate have to stay in phase with $\omega_{0n}t$ it is best to develop all frequencies around ω_{0n}, e.g.

$$\cos\omega_i t = \cos\delta\omega_i t \cdot \cos\omega_{0n} t - \sin\delta\omega_i t \cdot \sin\omega_{0n} t\,.$$

then, (8) can be written as

$$\begin{aligned}\mathbf{q} &= \mathbf{Q}[(\delta\mathbf{C}\mathbf{a} + \delta\mathbf{S}\mathbf{b})\cos\omega_{0n}t + (\delta\mathbf{C}\mathbf{b} - \delta\mathbf{S}\mathbf{a})\sin\omega_{0n}t] = \\ &= \mathbf{Q}[\mathbf{a}^*\cos\omega_{0n}t + \mathbf{b}^*\sin\omega_{0n}t] \end{aligned} \tag{15}$$

where $\delta\mathbf{C}$, $\delta\mathbf{S}$ are diagonal matrices of $\cos\delta\omega_i t$ and $\sin\delta\omega_i t$ respectively. $(\mathbf{Q}\mathbf{a}^*)_i$ is now the envelope signal in cell i which is relevant for particles in phase with $\cos\omega_{0n}t$. $(\mathbf{Q}\mathbf{b}^*)_i$ is a signal which decays and which rings with $\sin\omega_{0n}t$, i.e. it is out of phase with the particles.

4 Filling of the TESLA Cavity

As an example we choose the filling process of the superconducting cavity for the TESLA linear collider study. The cavity is a nine–cell, flat–tuned, π–mode structure. It is driven by a generator in the first cell. We assume that the beam induced voltage is half of the voltage generated by the driver. Then, the cavity voltage stays constant after the time $t_0 = \tau \ln 2$ when the beam is switched on. τ is the filling time

$$\tau = \frac{2Q_0}{\omega_\pi (1+\beta_1) {q_1^{(9)}}^2} = 0.832\,\mathrm{ms}$$

The parameters used are

$$\begin{aligned}
&f_\pi = f_{09} = 1.3\,\mathrm{GHz},\ Q_0 = 3\cdot 10^9,\ R/Q_0 = 112.3\,\Omega/\mathrm{cell} \\
&\beta_1 = 9\cdot 882 = 7938 \text{ generator coupling constant} \\
&L = 115.4\,\mathrm{mm} \text{ cell length} \\
&k = 0.0185 \text{ cell–to–cell coupling} \\
&T_b = 1\,\mu\mathrm{s} \text{ bunch distance},\ N_b = 800 \quad \#\,\text{bunches} \\
&N_e = 5\cdot 10^{10} \quad \#\, e^-/\text{bunch}
\end{aligned} \tag{16}$$

In Fig. 3a we see the envelopes in cells one and nine when cell one is shock excited by a voltage step. The time delay in cell nine of $\Delta T = 130$ ns corresponds exactly to the travelling time which a wave front needs to travel through the structure with a group velocity $v_g = 0.02c_0$ equal to the one in the center of the pass band, see Fig. 3b.

The normal filling of the cavity with a switched–on sine signal is shown in Fig. 4a. It shows the exponential increase of the cavity voltage and its flat top under beam loading. The curve is an overlay of the voltage envelopes in the first and the nineth cell. A zoom of the curve for very short times, Fig. 4b, and for the first and last bunches of the beam, Fig. 5, resolve the different signals in the cells. Evaluating the time delay between the filling of the 1st and 9th cell, see Fig. 4b, clearly proves again that the wavefront in an empty cavity travels with the average group velocity in the pass–band, i.e. with essentially the group velocity at the $\pi/2$–mode. The same is true for the refilling of the cavity when a bunch has taken out a certain amount of the energy. From Fig. 5 it can be seen that field levels are different in every cell and that the differences are larger at the beginning of the beam. But averaging over all cells results in a maximum voltage variation of only 0.5 $^o/_{oo}$ for the bunches. Finally, a study of the voltage sensitivity ΔV at the end of the beam against changes in the bunch charge ΔN_e gives $\Delta V/V \approx 0.4\,\Delta N_e/N_e$, i.e. the mean deviation in N_e has to be less than 2.5 times the allowable voltage variation.

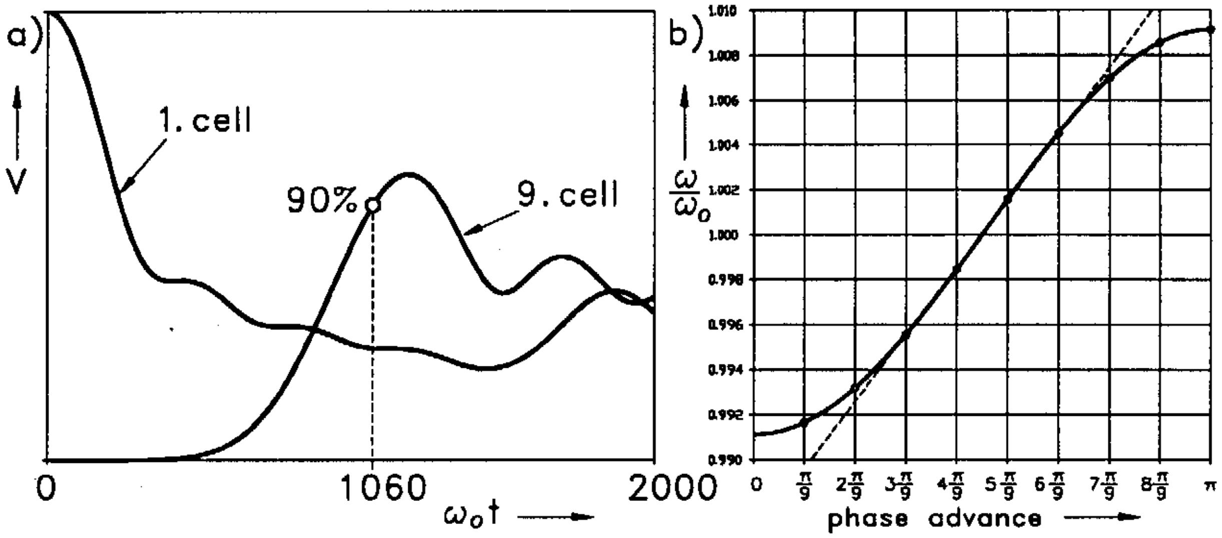

Figure 3: a) Envelopes in cell one and nine of the TESLA cavity for shock excitation of cell one. b) Dispersion digramm of the cavity.

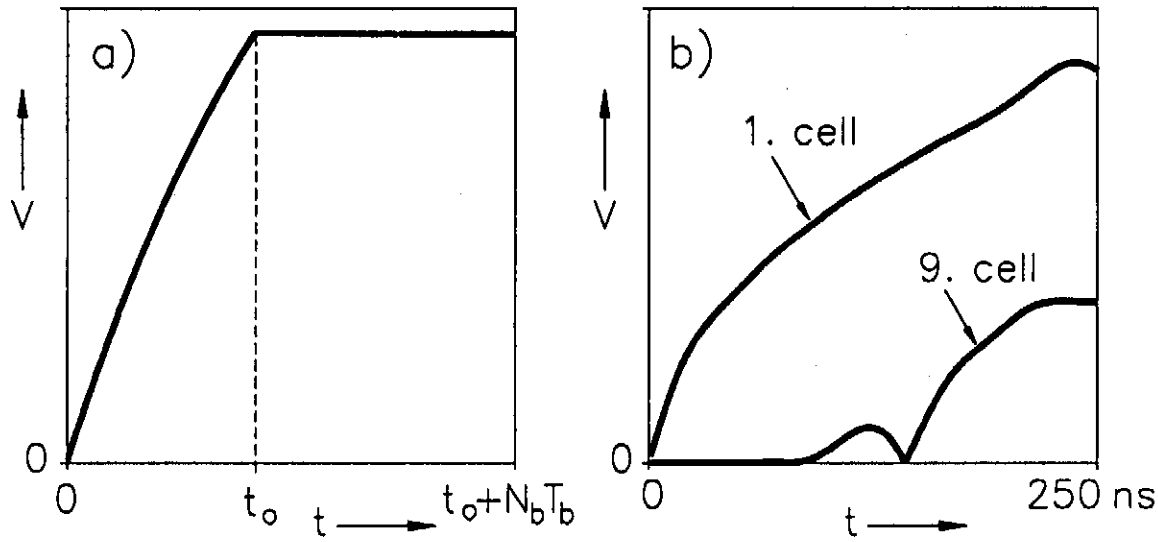

Figure 4: Filling process of the TESLA cavity with beam for $t > t_0$. a) Voltage envelopes in the 1st and 9st cell, b) zoom for small times.

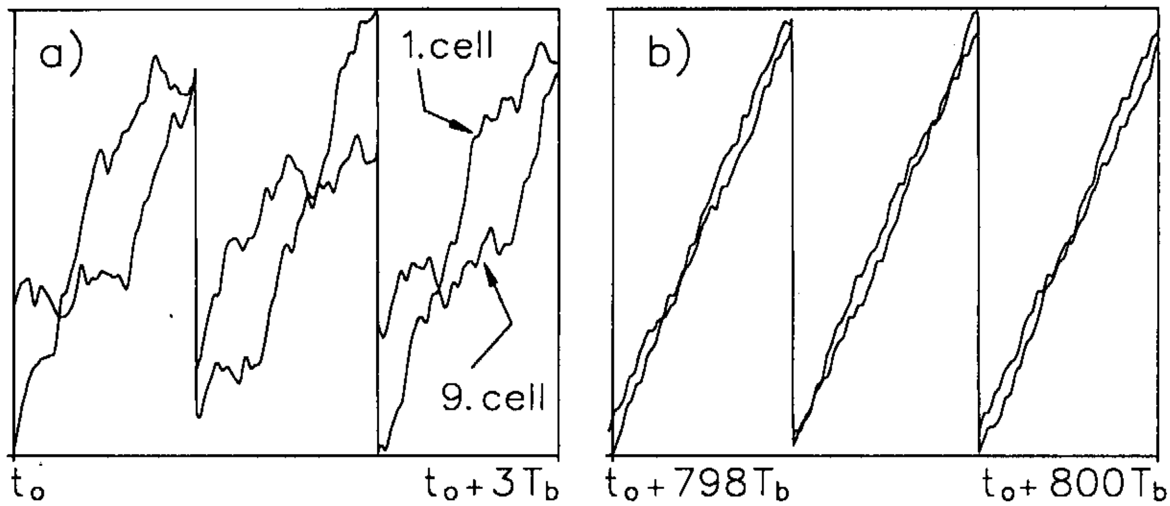

Figure 5: Blown–up curve of Fig. 4 for the a) first and b) last bunches of the beam.

Appendix

The eigenfrequencies ω_i and eigenvectors $\mathbf{q}_i$ of the steady–state, loss–free, homogenous system can easily be derived with standard matrix algebra as for instance treated in [2]. Starting with the loop equations for time–harmonic signals

$$(-\omega^2\mathbf{I}+\omega_0^2\mathbf{I}+\omega^2 k\mathbf{K})\mathbf{q}=0$$

we can write

$$\mathbf{M}\cdot\mathbf{q}=0 \tag{A1}$$

$$\mathbf{M}=\begin{bmatrix}\alpha & 1 & 0 & \cdots\\ 1 & \alpha & 1 & \cdots\\ 0 & 1 & \alpha & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix} \quad , \quad \alpha=\frac{1}{k}\left[\left(\frac{\omega_0}{\omega}\right)^2-1\right] \quad , \quad \omega_0=\frac{1}{\sqrt{LC}} \quad .$$

The eigensolutions ω_i, $\mathbf{q}^{(i)}$ of (A1) follow from $\det(\mathbf{M})=0$. If we call p_N the determinant of the $(N\times N)$ matrix $\mathbf{M}$ and p_k the determinant obtained from $\mathbf{M}$ after removing the last $N-k$ rows and columns we obtain

$$p_k=\begin{vmatrix}\alpha & 1 & 0 & \cdots\\ 1 & \alpha & 1 & \cdots\\ 0 & 1 & \alpha & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{vmatrix}_{k\times k} = \alpha p_{k-1}-p_{k-2} \quad . \tag{A2}$$

(A2) is a recursion formula for the determinant p_N with initial conditions $p_1=\alpha$, $p_2=\alpha^2-1$. Again, we can formulate (A2) in matrix notation

$$\begin{bmatrix}p_k\\ p_{k-1}\end{bmatrix}=\begin{bmatrix}\alpha & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}p_{k-1}\\ p_{k-2}\end{bmatrix}=\mathbf{P}\begin{bmatrix}p_{k-1}\\ p_{k-2}\end{bmatrix}=\mathbf{P}^{k-2}\begin{bmatrix}p_2\\ p_1\end{bmatrix} \quad . \tag{A3}$$

In order to perform $\mathbf{P}^{k-2}$ we transform $\mathbf{P}$ into diagonal form

$$\mathbf{D} = \mathbf{S}^{-1}\mathbf{P}\mathbf{S} \tag{A4}$$

where $\mathbf{D}$ is the diagonal matrix of the eigenvalues of $\mathbf{P}$

$$\lambda^2 - \alpha\lambda + 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = -\,\mathrm{e}^{\pm \mathrm{j}\varphi} \quad , \quad \alpha = -2\cos\varphi \tag{A5}$$

and $\mathbf{S}$ is the matrix of the column eigenvectors

$$\mathbf{S} = \begin{bmatrix} \mathrm{e}^{\mathrm{j}\varphi/2} & \mathrm{e}^{-\mathrm{j}\varphi/2} \\ -\mathrm{e}^{-\mathrm{j}\varphi/2} & -\mathrm{e}^{\mathrm{j}\varphi/2} \end{bmatrix} \quad . \tag{A6}$$

Having determined $\mathbf{D}$ and $\mathbf{S}$ we get from (A4)

$$\mathbf{P}^{k-2} = \mathbf{S}\mathbf{D}^{k-2}\mathbf{S}^{-1} = \frac{1}{\sin\varphi}\begin{bmatrix} -\sin(k-1)\varphi & -\sin(k-2)\varphi \\ \sin(k-2)\varphi & \sin(k-3)\varphi \end{bmatrix} \quad . \tag{A7}$$

Substituting (A7) into (A3) it follows for $k = N$

$$p_N = -\frac{\sin(N+1)\varphi}{\sin\varphi} \tag{A8}$$

with zeros for

$$\varphi_i = \frac{i\pi}{N+1} \quad , \quad i = 1, 2, \ldots, N \tag{A9}$$

and therefore for

$$\alpha_i = \frac{1}{k}\left[\left(\frac{\omega_0}{\omega_i}\right)^2 - 1\right] = -2\cos\varphi_i$$

$$\omega_i = \frac{\omega_0}{\sqrt{1 - 2k\cos\varphi_i}} \quad , \quad \varphi_i = \frac{i\pi}{N+1} \quad , \quad i = 1, 2, \ldots, N \quad . \tag{A10}$$

The eigenvectors can be calculated in a similar way. Writing one line of (A1) we obtain again a recursion formula

$$q_k^{(i)} = -\alpha q_{k-1}^{(i)} - q_{k-2}^{(i)} \quad , \quad k = 3, \ldots, N$$

which can be written

$$\begin{bmatrix} q_k^{(i)} \\ q_{k-1}^{(i)} \end{bmatrix} = \begin{bmatrix} -\alpha_i & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{k-1}^{(i)} \\ q_{k-2}^{(i)} \end{bmatrix} = \mathbf{Q} \begin{bmatrix} q_{k-1}^{(i)} \\ q_{k-2}^{(i)} \end{bmatrix} = \mathbf{Q}^{k-2} \begin{bmatrix} q_2^{(i)} \\ q_1^{(i)} \end{bmatrix} \tag{A11}$$

with initial conditions $q_1^{(i)} = c^{(i)}$, $q_2^{(i)} = -\alpha_i c^{(i)}$. Now, we proceed in the same way as for the calculation of the eigenvalues and find

$$q_k^{(i)} = c^{(i)}\frac{\sin k\varphi_i}{\sin\varphi_i} \quad , \quad k = 3, \ldots, N \quad .$$

The constant $c^{(i)}$ we choose such that $\mathbf{q}^{(i)t} \cdot \mathbf{q}^{(i)}$ is normalized to one and obtain

$$q_k^{(i)} = \sin(k\varphi_i) \Big/ \sqrt{\sum_k \sin^2(k\varphi_i)} \quad , \quad \varphi_i = \frac{i\pi}{N+1} \quad , \quad i,k = 1,\ldots,N \quad . \tag{A12}$$

(A10) and (A12) give the steady–state eigenfrequencies and eigenvectors of a string of N loss–free resonators.

As can be checked easily the amplitudes (A12) of every mode i are unevenly distributed over the resonators. But often, as for our example of the TESLA cavity, every resonator represents an accelerating cell of an RF structure and one normally wants equal amplitudes and a π phase–shift from cell to cell. This is called a flat–tuned π–mode structure. The flat tuning is achieved by tuning for instance the inductance of the end–cells such that $q_i = -q_{i-1}$. From the loop equations

$$\left[1 + \frac{\Delta L}{L} - \left(\frac{\omega_0}{\omega_N}\right)^2\right] q_1 - kq_2 = 0$$

$$-kq_1 + \left[1 - \left(\frac{\omega_0}{\omega_N}\right)^2\right] q_2 - kq_3 = 0$$

follows then

$$\frac{\Delta L}{L} = k \tag{A13}$$

in order to fulfill $q_i = -q_{i-1}$. Deviding the loop equations by $-k$ we end up with the system matrix of the flat–tuned π–mode equal to

$$\mathbf{M}_\pi = \begin{bmatrix} \alpha - 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha & 1 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & 1 \\ 0 & 0 & 0 & \cdots & 1 & \alpha - 1 \end{bmatrix} \quad , \quad \alpha = \frac{1}{k}\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right] \quad . \tag{A14}$$

Eigenfrequencies and eignvectors are now found in the same way as above. The only difference beeing in the first and last row of $\mathbf{M}_\pi$ which have to be treated separately. Then the determinant of $\mathbf{M}_\pi$ becomes

$$\det(\mathbf{M}_\pi) = -(2-\alpha)p_{N-1} = -2(1+\cos\varphi)\frac{\sin N\varphi}{\sin\varphi} \tag{A15}$$

yielding eigenfrequencies of

$$\varphi_i = i\frac{\pi}{N} \quad , \quad i = 1,\ldots,N$$

$$\omega_i = \frac{\omega_0}{\sqrt{1 - 2k\cos\varphi_i}} \quad . \tag{A16}$$

For the eigenvectors we obtain

$$\begin{bmatrix} q_k^{(i)} \\ q_{k-1}^{(i)} \end{bmatrix} = \frac{c^{(i)}}{\sin\varphi} \begin{bmatrix} \sin(k-1)\varphi & -\sin(k-2)\varphi \\ \sin(k-2)\varphi & -\sin(k-3)\varphi \end{bmatrix} \begin{bmatrix} 1+2\cos\varphi \\ 1 \end{bmatrix}$$

or

$$q_k^{(i)} = c^{(i)} 2 \frac{\cos\varphi_i/2}{\sin\varphi_i} \sin\left(k-\frac{1}{2}\right)\varphi_i$$

and after normalization

$$q_k^{(i)} = \sin\left[\left(k-\frac{1}{2}\right)\varphi_i\right] \Big/ \sqrt{\sum_k \sin^2\left[\left(k-\frac{1}{2}\right)\varphi_i\right]} \quad . \tag{A17}$$

References

[1] M. F. Gardner and J. L. Barnes, Transients in linear systems. Wiley, New York 1942.

[2] M. C. Pease, Methods of matrix algebra. Academic Press 1965.