

Emittance Dilution in the TESLA FEL Second Bunch Compression System

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Abstract.

This paper presents developed results of investigation of the transverse emittance dilution in the TESLA FEL bunch compression system. The calculations for the quadrupole and sextupole field component of the bending magnet are performed. The requirements for the bending magnet field of the dipole magnets of the 2nd bunch compressor are determined.

The second bunch compression system for the TESLA Test Facility FEL consists of four rectangular magnets with adjusting focusing system. The bunch compressor has to decrease the bunch longitudinal rms size from 0.8 mm to 0.2 mm at an energy of 144 MeV [1].

In this report the main parameters of the second bunch compression system are determined to get the required compression. The transverse emittance dilution due to the quadrupole and sextupole components of the bending magnet field in the bunch compression system is calculated for the energy dispersion about 0.5 %. The main requirements for the bending magnets of the 2nd bunch compression system are determined.

Introduction.

In the TTF bunch compression systems the C-type dipole magnets will be used. For this type of magnet field can be presented as combination of dipole, quadrupole and sextupole components. As is shown in the report [2] the quadrupole (sextupole) field component in dipole magnets of the bunch compression system will dilute the horizontal emittance. First of all it is necessary to investigate influence of the quadrupole magnet field component on beam emittance growth.

1. Analytical approach.

The main peculiarity of the bunch compression system is the following. Each particle of the bunch moving in transverse magnet field spends time determined by own energy and bending magnet field value. The magnet field is not uniform in a working region due to magnet field errors which can be presented as some quadrupole and sextupole magnet field components. In this case calculation of the time should be made accurately to investigate longitudinal and transverse particle motions.

Let's consider the particle motion in the Cartesian coordinate system $\{x,y,z\}$. The dipole magnet field has the dipole and quadrupole components and one can present this field as $\{B_x, B_y, B_z\}$, where $B_x = gyB_0$, $B_y = B_0(1+gx)$, $B_z = 0$, $g = G/B_0$.

$G = \partial B_y / \partial x$. Using these assumption the particle motion equations can be written in the following form

$$\begin{aligned}
 x'' &= \alpha (1 + gx)z' , \\
 z'' &= -\alpha (1 + gx)x' + \alpha gyy' , \\
 y'' &= -\alpha gyz' , \\
 \alpha &= \frac{|e|B_0}{m_0\gamma c^2}
 \end{aligned} \tag{1}$$

where all values are functions of the "time" $s = ct$, c - the light velocity.

To investigate the emittance dilution due to the non-uniformity of the bending magnet field let's consider coordinates of an arbitrary particle of the bunch as some combination

$$\begin{aligned}
 x &= x_r + \xi + \sigma , \\
 y &= y_r + \phi + \lambda , \\
 z &= z_r + \eta + \varepsilon ,
 \end{aligned} \tag{2}$$

where (x_r, y_r, z_r) are the coordinates of the "reference" particle, (ξ, ϕ, η) are the additional correction of particle coordinates, connected with real particle energy, $(\sigma, \lambda, \varepsilon)$ are the additional correction of particle coordinates, connected with the quadrupole gradient of the real bending magnet field. The "reference" particle is the particle with some fixed energy (for the second TTF bunch compressor this energy is 144MeV). Using this approach the analytical solutions of the particle motion equations are defined.

The solutions of the equations (1) for the "reference" particle in the case of uniform magnet field are

$$\begin{aligned}
 x_r &= x_0 + x'_0 \frac{\sin(\alpha_r s)}{\alpha_r} + z'_0 \frac{1 - \cos(\alpha_r s)}{\alpha_r} , \\
 y_r &= y_0 + y'_0 s , \\
 z_r &= z_0 + z'_0 \frac{\sin(\alpha_r s)}{\alpha_r} - x'_0 \frac{1 - \cos(\alpha_r s)}{\alpha_r} ,
 \end{aligned} \tag{3}$$

where $(x_0, y_0, z_0), (x'_0, y'_0, z'_0)$ are the initial coordinates and velocities of the "reference" particle, $\alpha_r = \frac{|e|B_0}{m_0\gamma_r c^2}$ is proportional to the inverse value of the curvature radius of the "reference" particle in the ideal bending magnet field B_0 , $\gamma_r = 1 + E_r / (m_0 c^2)$ is the relativistic factor of the "reference" particle, E_r is the energy of this particle. Using these solutions one can define the coordinates of an arbitrary particle in the case on the uniform bending magnet field

$$\begin{aligned} \xi = & \xi_0 + \frac{\xi'_0}{\alpha} \sin(\alpha s) + \frac{\eta'_0}{\alpha} [1 - \cos(\alpha s)] + z'_0 \left(\frac{1}{\alpha} - \frac{1}{\alpha_r} \right) + \\ & + z'_0 \left[\frac{\cos(\alpha_r s)}{\alpha_r} - \frac{\cos(\alpha s)}{\alpha} \right] + x'_0 \left[\frac{\sin(\alpha s)}{\alpha} - \frac{\sin(\alpha_r s)}{\alpha_r} \right] \end{aligned} \quad (4)$$

$$\begin{aligned} \xi' = & \xi'_0 \cos(\alpha s) + \eta'_0 \sin(\alpha s) + x'_0 [\cos(\alpha s) - \cos(\alpha_r s)] + \\ & + z'_0 [\sin(\alpha s) - \sin(\alpha_r s)] \end{aligned}$$

$$\begin{aligned} \eta = & \eta_0 + \frac{\eta'_0}{\alpha} \sin(\alpha s) - \frac{\xi'_0}{\alpha} [1 - \cos(\alpha s)] - x'_0 \left(\frac{1}{\alpha} - \frac{1}{\alpha_r} \right) - \\ & - x'_0 \left[\frac{\cos(\alpha_r s)}{\alpha_r} - \frac{\cos(\alpha s)}{\alpha} \right] + z'_0 \left[\frac{\sin(\alpha s)}{\alpha} - \frac{\sin(\alpha_r s)}{\alpha_r} \right] \end{aligned}$$

$$\begin{aligned} \eta' = & \eta'_0 \cos(\alpha s) - \xi'_0 \sin(\alpha s) + \\ & + x'_0 [\sin(\alpha_r s) - \sin(\alpha s)] - z'_0 [\cos(\alpha_r s) - \cos(\alpha s)] \end{aligned}$$

$$\phi = \phi_0 + \phi'_0 s$$

The initial values $(\xi_0, \xi'_0, \eta_0, \eta'_0, \phi_0, \phi'_0)$ are determined for each particle of the bunch from distribution in the 6th dimensional ellipsoid before injection into the first magnet of the bunch compression system.

To simplify the solution of the general equations (1) in the case of non-uniform unhomogeneous bending magnet field one can consider only values proportional to the first order of g , because $g \propto 10^{-4}$. Using this assumption one can get the equations on

the additional part of the particle coordinates defined by some quadrupole coefficient of the magnet field and the own particle energy

$$\begin{aligned}
 \sigma' &= \alpha \varepsilon' + g \alpha z'_r x_r \\
 \varepsilon'' &= -\alpha \sigma' - g \alpha x_r x'_r + g \alpha y_r y'_r \\
 \lambda'' &= -g \alpha y_r z'_r
 \end{aligned} \tag{5}$$

Solutions of these equations may be written in the form

$$\begin{aligned}
 \sigma &= \frac{D_0}{\alpha^2} [1 - \cos(\alpha s)] + \frac{D_1}{\alpha} \left[s - \frac{\sin(\alpha s)}{\alpha} \right] - \\
 &- 2 \frac{D_2}{\alpha^3} \left[1 - \cos(\alpha s) - \frac{(\alpha s)^2}{2} \right] + \sum_n [a_n U_n(s) + b_n V_n(s)]
 \end{aligned} \tag{6}$$

$$\begin{aligned}
 \sigma' &= \frac{D_0}{\alpha} \sin(\alpha s) + \frac{D_1}{\alpha} [1 - \cos(\alpha s)] - 2 \frac{D_2}{\alpha^2} [\sin(\alpha s) - \alpha s] + \\
 &+ \sum_n (n \alpha_r) \left\{ -a_n \left[V_n(s) + \frac{1 - \cos(\alpha s)}{\alpha} \right] + b_n \left[U_n(s) + \frac{\sin(\alpha s)}{\alpha} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon &= \frac{D_1}{\alpha^2} [1 - \cos(\alpha s)] + \frac{1}{\alpha} \left(D_0 - 2 \frac{D_2}{\alpha} \right) \left[\frac{\sin(\alpha s)}{\alpha} - s \right] + \\
 &+ \sum_n [-a_n V_n(s) + b_n U_n(s)]
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon' &= \frac{D_1}{\alpha} \sin(\alpha s) - \frac{1}{\alpha} \left(D_0 - 2 \frac{D_2}{\alpha} \right) [1 - \cos(\alpha s)] - \\
 &- \sum_n (n \alpha_r) \left\{ a_n \left[U_n(s) + \frac{\sin(\alpha s)}{\alpha} \right] + b_n \left[V_n(s) + \frac{1 - \cos(\alpha s)}{\alpha} \right] \right\}
 \end{aligned}$$

where $n=1,2$
$$U_n(s) = 2 \frac{\sin\left(\frac{n\alpha_r - \alpha}{2}s\right)}{n\alpha_r - \alpha} \cos\left(\frac{n\alpha_r + \alpha}{2}s\right) - \frac{1}{\alpha} \sin(\alpha s)$$

$$V_n(s) = 2 \frac{\sin\left(\frac{n\alpha_r - \alpha}{2}s\right)}{n\alpha_r - \alpha} \sin\left(\frac{n\alpha_r + \alpha}{2}s\right) - \frac{1 - \cos(\alpha s)}{\alpha}$$

a_n, b_n, D_0, D_n are values related with the initial conditions for the reference particle, the uniform value of the bending magnet field, the quadrupole coefficient of the magnet field and the energy of each particle

$$a_1 = g \frac{\alpha}{\alpha_r} x'_0 \left(x_0 + \frac{z'_0}{\alpha_r} \right)$$

$$a_2 = -g \frac{\alpha}{2\alpha_r^2} x'_0 z'_0$$

$$b_1 = g \frac{\alpha}{\alpha_r} z'_0 \left(x'_0 + \frac{z'_0}{\alpha_r} \right)$$

$$b_2 = g \frac{\alpha}{4\alpha_r^2} (x_0'^2 - z_0'^2)$$

$$D_0 = -g \frac{\alpha}{2\alpha_r} (x_0'^2 + z_0'^2)$$

$$D_1 = g\alpha y_0 y'_0$$

$$D_2 = \frac{1}{2} g\alpha y_0'^2$$

The solution for the y-direction one can write in the following form

$$\begin{aligned} \lambda(s) = & g \frac{\alpha}{\alpha_r} \left(\frac{y_0 z'_0}{\alpha_r} - \frac{2y'_0 x'_0}{\alpha_r^2} \right) \cos(\alpha_r s) - \left(y_0 x'_0 + \frac{2y'_0 z'_0}{\alpha_r^2} \right) \sin(\alpha_r s) + \\ & + y'_0 \frac{z'_0}{\alpha_r} \cdot s \cdot \cos(\alpha_r s) - y'_0 \frac{x'_0}{\alpha_r} \cdot s \cdot \sin(\alpha_r s) + C_1 \cdot s + C_2 \end{aligned}$$

where $C_1 = g \frac{\alpha}{\alpha_r} \left(y_0 x'_0 + \frac{y'_0 z'_0}{\alpha_r} \right)$, $C_2 = g \frac{\alpha}{\alpha_r} \left(-2 \frac{x'_0 y'_0}{\alpha_r^2} + y_0 \frac{z'_0}{\alpha_r} \right)$. Using these solutions the calculation of the particle motion in the magnets of the bunch compression system is

performed and the emittance dilution due to the quadrupole coefficient of the bending magnet field is estimated.

To define the beam emittance one can use the general definition of the rms beam emittance

$$\epsilon^2 = \sum_i (x_i - \bar{x})^2 / N \cdot \sum_i (v_i - \bar{v})^2 / N - \sum_i [(x_i - \bar{x})(v_i - \bar{v})]^2 / N, \quad (7)$$

where x_i, v_i are the coordinates and velocities of i -particle, \bar{x}, \bar{v} are the average values.

To get particle distribution in the bunch before the TTF bunch compression system one can use the microcanonical distribution in the transverse phase ellipses with some defined half-axis's and the Gaussian longitudinal distribution with special arrangement along the bunch to get a "linear" correlation between the particle kinetic energy and longitudinal position (see **Appendix**). The special arrangement is used to investigate the longitudinal bunch compression.

2. Longitudinal particle motion in the 2nd TTF bunch compression system.

The main dimensions of the second bunch compressor are the following. The dipole magnet length is 50 cm, the distances between the first and second magnets and the third and fourth magnets are 50 cm, the distance between the second and third magnets is 100 cm. The energy of the "reference" particle is 144MeV, the initial longitudinal rms bunch size is 0.08cm and the energy dispersion is 0.005. The bending angles are changed between 18 degrees and 23 degrees [1]. The corresponding dipole magnet field components are 2968.617 Gs and 3753.616 Gs respectively. The working region length of the dipole magnet in the x-direction in this case should be 18.5 cm. The reference particle trajectories for these values of the bending magnet field are presented in **Fig.1**.

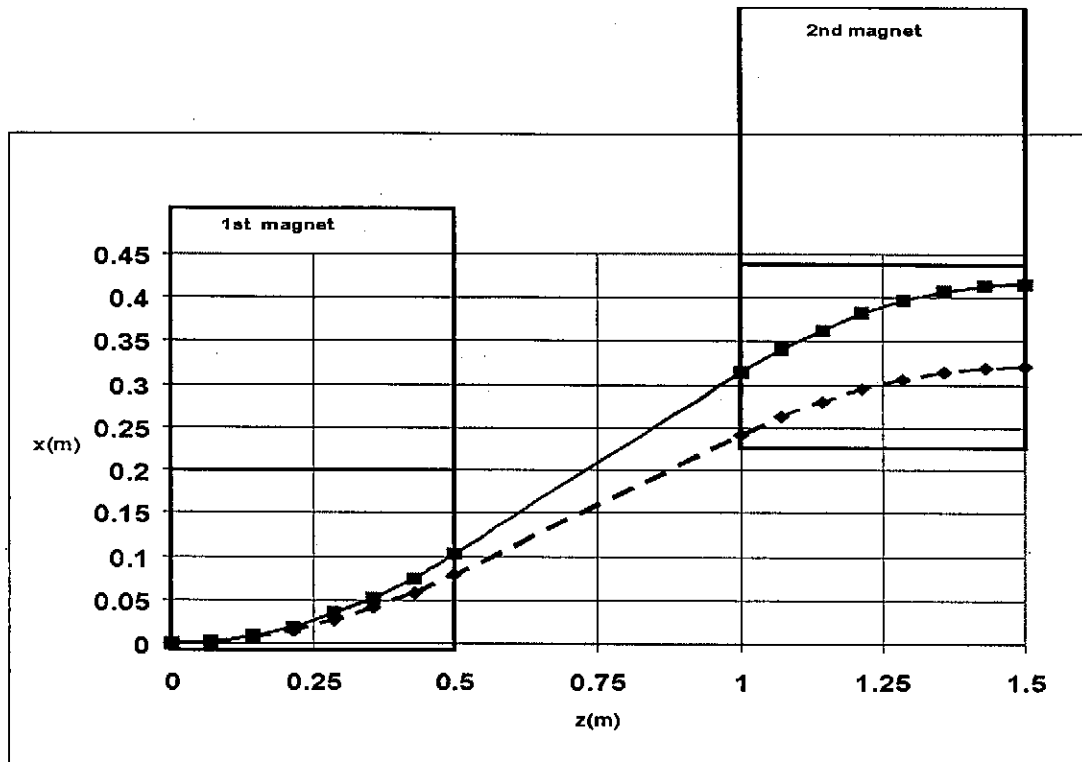


Fig.1

As a first step the longitudinal particle motion in the bunch compression system is considered. Using the developed program the bunch compression without space charge effects (for the “cold” beam) in the magnet chicane is analyzed. In **Fig.2** the relative longitudinal bunch size as a function of the bending magnet field is shown to demonstrate the longitudinal bunch size compression.

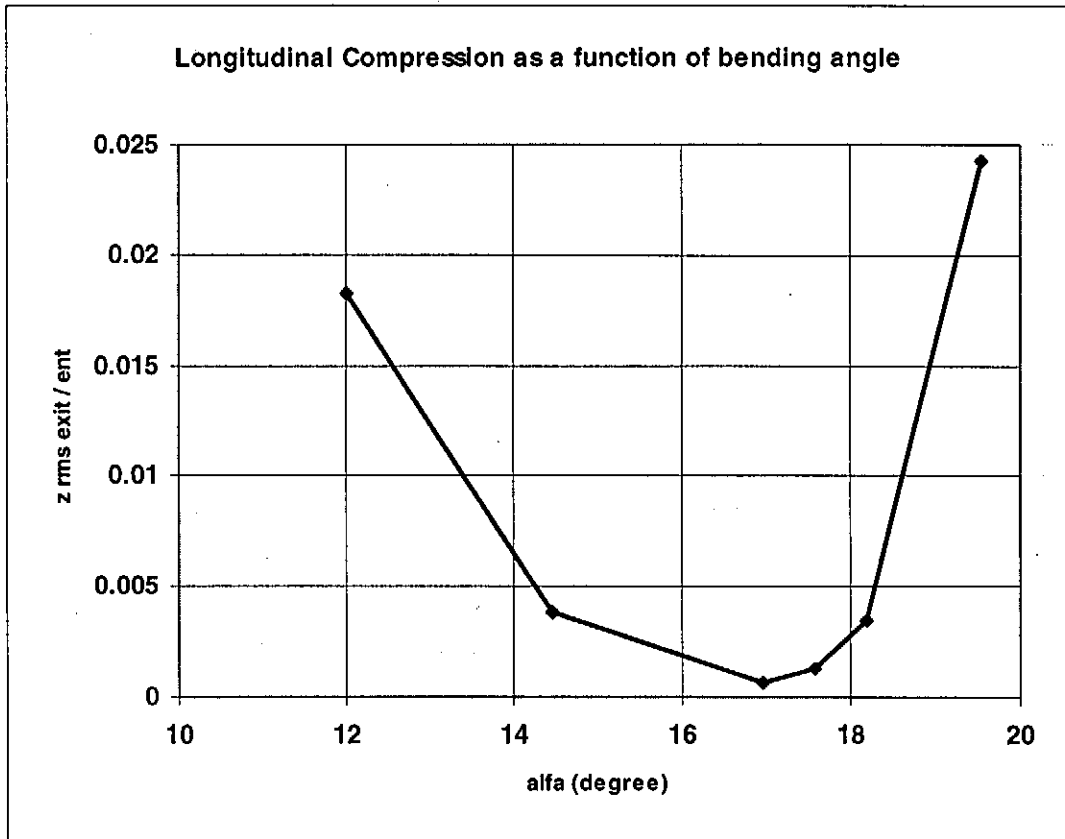


Fig.2

3. Transverse emittance dilution in the bunch compression system.

The transverse emittance dilution may be estimated by the PARMELA program. Obviously that the transverse emittance for the system with an "ideal" bending magnet field is constant before and after the bunch compressor. Using the PARMELA program one can get this result with 1...2% accuracy.

Using the described algorithm the special program is made to investigate emittance dilution due to quadrupole magnet field component in the bending magnets of the second TESLA FEL bunch compression system. This results is checked by the PARMELA program. And as the next step, the transverse emittance dilution due to

sextupole component of the bending magnet field is estimated by the PARMELA program.

3.1 Transverse emittance dilution due to quadrupole magnet field component.

To estimate transverse emittance dilution due to quadrupole component of the bending magnet field of the 2nd bunch compression system one can use the developed program. The particle bunch can be presented as some slices in the transverse direction. Each slice has the own energy. The energy of the slices are in the range 144 ± 0.720 MeV ($\sigma = 0.005$). The transverse coordinates and velocities of the particles in the slice are generated to get the microcanonical distribution. Before and after the bunch compression system the transverse emittance is determined using the general definition (7) for all particles of the bunch.

The initial bunch parameters are the following:

$\sigma_z = 0.08(\text{cm})$, $\sigma_E = 0.005$, $\epsilon^n = 1 \cdot (\pi) \cdot \text{mm} \cdot \text{mrad}$ or $\epsilon^n = 2 \cdot (\pi) \cdot \text{mm} \cdot \text{mrad}$, $\beta_x = 15.0\text{m}$, $\beta_y = 8.0\text{m}$, $\alpha_x = \alpha_y = 0.\text{rad}$, $E_r = 144\text{MeV}$, where σ_z are the rms longitudinal beam size, σ_E is the energy dispersion, ϵ^n is the normalized transverse emittance.

Fig.3 and **Fig.4** show the relative emittance dilution $D\epsilon_x = (\epsilon^{\text{exit}} - \epsilon^{\text{initial}}) / \epsilon^{\text{initial}}$ as a function of the average quadrupole component in the working region of the dipole magnet

$\langle g \rangle = \left\langle \frac{1}{B_0} \frac{\partial B_y}{\partial x} \right\rangle$ for the dipole magnet field 2968.617 Gs and 3753.616 Gs for different initial emittance.

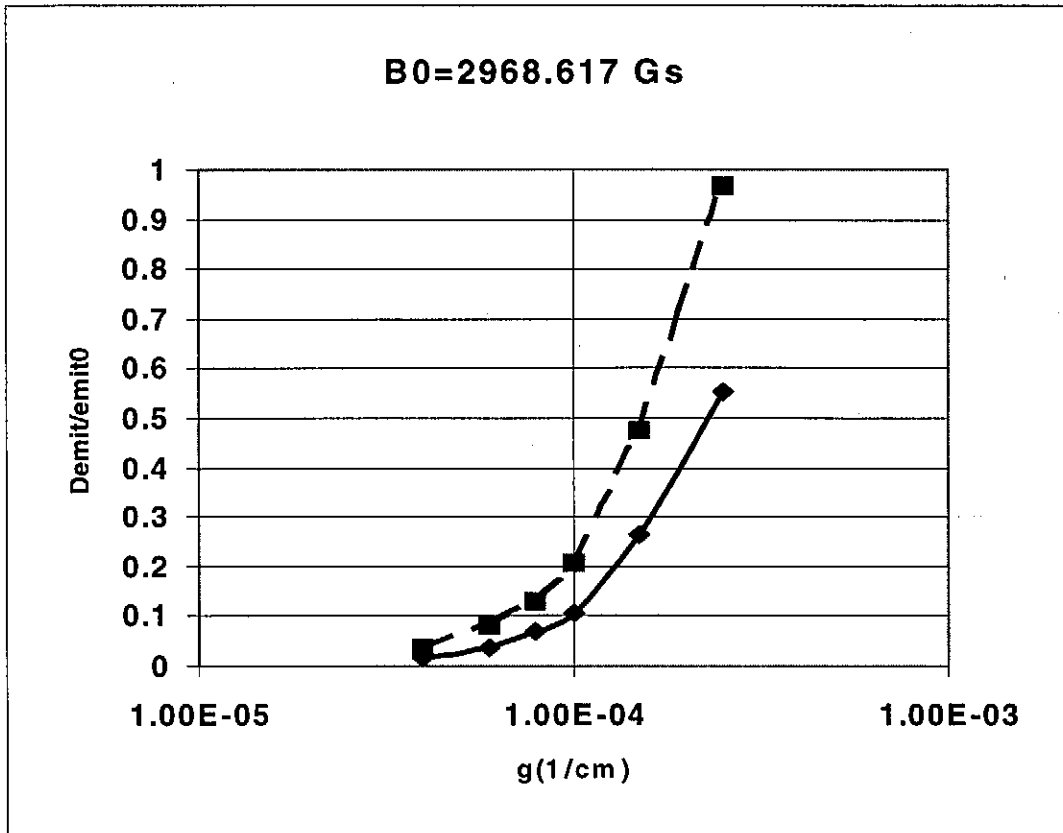


Fig.3

Fig.3 presents the relative emittance dilution as a function of the quadrupole component of the bending magnet field for different values of the normalized initial transverse emittance : 1π mm.mrad- the dashed line, 2π mm.mrad - the solid line. The dipole magnet field is 2968.617 Gs.

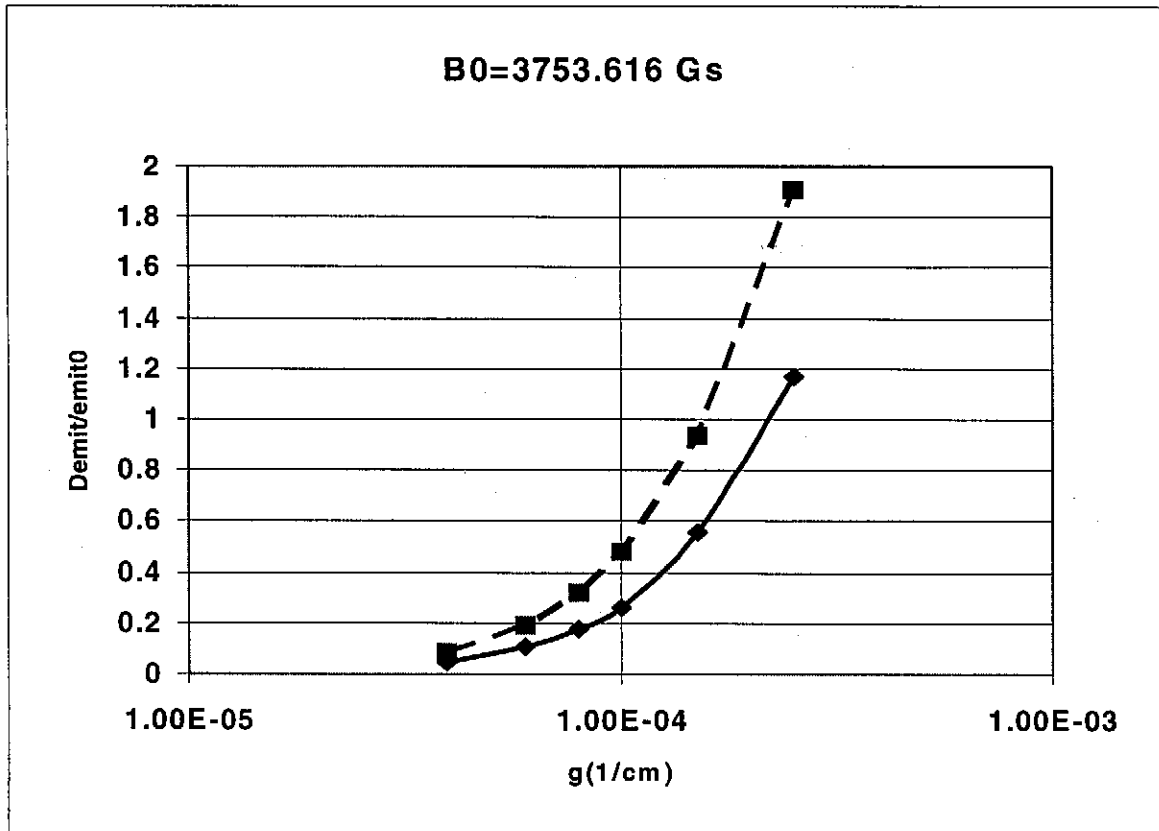
**Fig.4**

Fig.4 presents the relative emittance dilution as a function of the quadrupole component of the bending magnet field for different values of the normalized initial transverse emittance : 1π mm.mrad - the dashed line, 2π mm.mrad - the solid line. The dipole magnet field is 3753.616 Gs.

Fig.5 shows the comparison of the results of the emittance growth calculation due to quadrupole field component for the dipole magnet field value 3200Gs corresponding to the 20 degrees bending angle. To analyse the transverse emittance dilution using the PARMELA program particles of the bunch are generated randomly in a four dimensional transverse hyperspace with uniform phase and random energy (INPUT6). The initial parameters are the same as mentioned above. The obtained result corresponds to the emittance with the 90% of the particles. To present the bunch 1000 particles is used. To include the quadrupole field component in the bending magnet the magnet can be presented as a few "short" bending magnets.

Between the “short” magnets the quadrupole and sextupole lenses can be installed to model the “real” magnet field of the dipole magnet with dipole, quadrupole and sextupole components. Length of the lens should be small so that the emittance behavior does not change due to additional drift space.

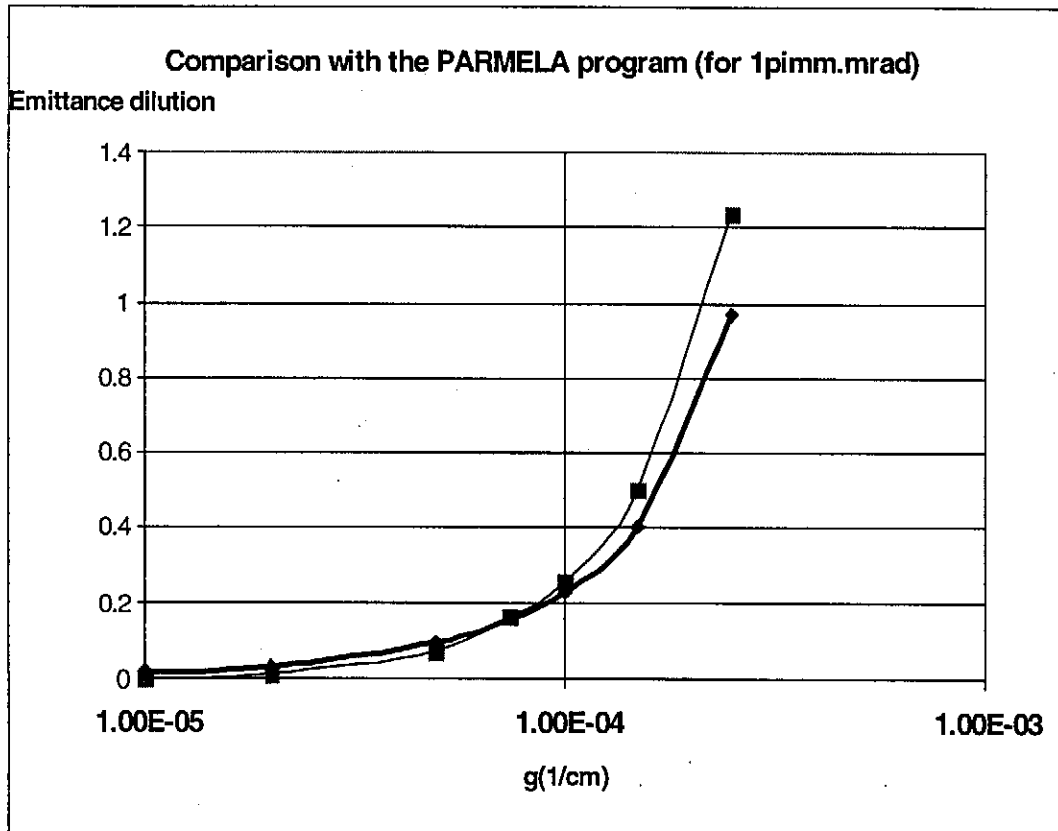


Fig.5

As one can see from obtained results to get the **10%** dilution of the x-emittance in the 2nd TTF bunch compression system with the dipole magnet field between 2968.617 Gs and 3753.616 Gs the average quadrupole component of the bending magnet field in the working region should be:

7.0e-5 (1/cm) for the 1pi.mm.mrad initial transverse normalized emittance and 2968.617 Gs magnet field;

1.0e-4 (1/cm) for the 2pi.mm.mrad initial transverse normalized emittance and 2968.617 Gs magnet field;

4.0e-5 (1/cm) for the 1pi.mm.mrad initial transverse normalized emittance and 3753.616 Gs magnet field;

6.0e-5 (1/cm) for the 2pi.mm.mrad initial transverse normalized emittance and 3753.616 Gs magnet field.

3.2 Transverse emittance dilution due to sextupole magnet field component.

To estimate the emittance dilution due to the sextupole magnet field component in the bending magnet the PARMELA program is used. In this case the quadrupole component is fixed. **Fig.6** demonstrate the relative emittance dilution as a function of the sextupole field component of the bending magnet field for the dipole magnet field value 3200 Gs. The quadrupole field component is **$g=5.0e-5$ (1/cm)** that corresponds to the **9.5%** emittance growth of the **1pi mm.mrad** beam without the sextupole field component . To estimate the difference of the sextupole component for the another initial condition the extra calculation is made for the **2pi mm.mrad** beam with **$g=7.5e-5$ (1/cm)** that corresponds to **8.5%** emittance growth.

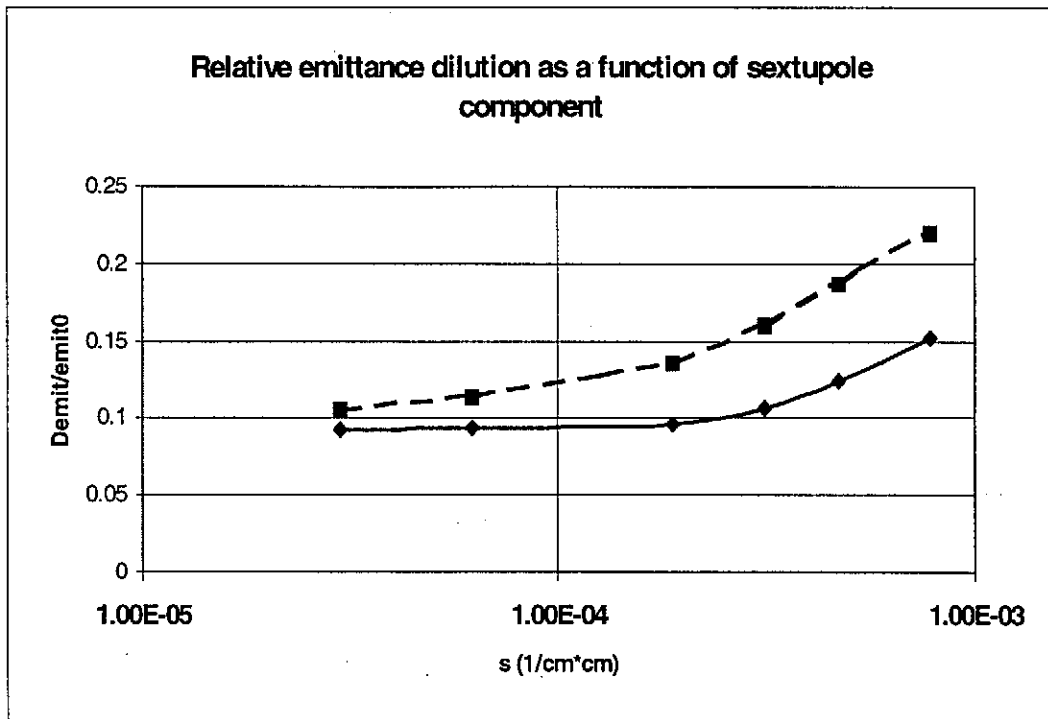


Fig.6

One can see from Fig.6 for the average sextupole field component of the bending magnet field about $(2..5)e-4$ (1/cm*cm) the relative emittance dilution is less than 15% in the case of some quadrupole field components, as is indicated above.

4. Magnet field requirements .

The main parameters on the dipole magnet of the second bunch compression system are collected in the Table 1.

Table 1.

**Main requirements to the dipole magnets
of the second TTF bunch compression system**

(C-type dipole magnet)

Bending angle for the reference particle with the energy 144MeV (degree)	18.0...23.0
Length of the bending magnet (cm)	50.0
Gap size (cm)	≥ 2.0
Length of the working region (cm)	≥ 18.5
Dipole component of the magnet field , B0 (Gs)	2968.617...3753.616
Average quadrupole component of the bending magnet field in the working region to provide the transverse emittance dilution about 10% without sextupole component $g = \left\langle \frac{1}{B_0} \cdot \frac{\partial B_y}{\partial x} \right\rangle \left(\frac{1}{\text{cm}} \right)$	$\sim 7.5 \cdot 10^{-5}$ (1pi / 2968.617Gs) $\sim 1.0 \cdot 10^{-4}$ (2pi / 2968.617 Gs) $\sim 4.0 \cdot 10^{-5}$ (1pi/ 3753.616 Gs) $\sim 6.0 \cdot 10^{-5}$ (2pi/ 3753.616 Gs)
Average sextupole component of the bending magnet field in the working region to produce the transverse emittance dilution about 15% for fixed g-values $s = \left\langle \frac{1}{B_0} \cdot \frac{\partial^2 B_y}{\partial x^2} \right\rangle \left(\frac{1}{\text{cm}^2} \right)$	$\sim 5 \cdot 10^{-4}$ (1 pi mm.mrad) $\sim 2 \cdot 10^{-4}$ (2 pi mm.mrad)

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References:

1. T.Limberg, H.Weise, A.Molodozhentsev, V.Petrov The bunch compression system at the TESLA Test Facility FEL, DESY PRINT, September 1995, TESLA-FEL 95-04.
2. P.Emma Bunch Compressor Beamlines for the TESLA and S-Band Linear Collider, DESY PRINT, July 1995, TESLA-FEL 95-17.
3. I.M.Kapchinsky, V.V.Vladimirsky Proc. of the Intern. Conference on High Energy Acceler. and Instrum., CERN, Geneva, 1959, p.274.

Appendix.

Particle generator.

Initial conditions for the bunch particles are defined to present the stationary system with the following distribution function in the 4th dimensional phase space

$$f(x, x', y, y') = A \cdot \varphi \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{x'^2}{c^2} + \frac{y'^2}{d^2} \right) \quad (1.1)$$

where (x, x', y, y') - the transverse coordinates and velocities of the particles in the bunch, $(a, c; b, d)$ - the half-axis of the phase ellipses.

To generate the particles with required initial bunch sizes it is necessary

1. to put the bunch particles in the 4th dimensional phase space in according with the distribution function (1.1) and
2. to provide in the longitudinal direction a "linear" correlation between particle energy and particle longitudinal position.

Let's use the polar coordinates to generate the particles in the 4th dimensional phase ellipse. In this case one can write the relations

$$\begin{aligned} x &= a \cdot r \cdot \cos(\theta_1) \\ y &= b \cdot r \cdot \sin(\theta_1) \cdot \cos(\theta_2) \\ x' &= c \cdot r \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\theta_3) \\ y' &= d \cdot r \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \end{aligned} \quad (1.2)$$

where $0 \leq r \leq \infty$, $0 \leq \theta_i \leq \pi$ ($i = 1, 2$), $0 \leq \theta_3 \leq 2\pi$, r, θ_i ($i = 1, 2, 3$) - the polar coordinates of the particle.

In these coordinates the normalized distribution function is a function of r

$$f(r) = \frac{\varphi(r^2)}{2\pi^2 abcd \int_0^{\infty} \varphi(r^2) r^3 dr} \quad (1.3)$$

and an element of the 4th dimensional phase volume can be presented as

$$dV = a \cdot b \cdot c \cdot d \cdot r^3 \cdot \sin^2(\theta_1) \cdot \sin(\theta_2) \cdot dr \cdot d\theta_1 \cdot d\theta_2 \cdot d\theta_3 \quad (1.4)$$

Number of particles in an elementary cell with coordinates $(r_i, r_i + \Delta r), (\theta_{ki}, \theta_{ki} + \Delta\theta_k), k = 1, 2, 3$ one can define as

$$dN_i = \frac{M}{\pi^2} \left\{ \left[\Delta\theta_1 - \sin(\Delta\theta_1) \cdot \cos(2\theta_{1i} + \Delta\theta_1) \right] \cdot \sin\left(\theta_{2i} + \frac{\Delta\theta_2}{2}\right) \cdot \sin\left(\frac{\Delta\theta_2}{2} \Delta\theta_3\right) \right\} \times$$

$$\times \frac{\int_{r_i}^{r_i + \Delta r} \varphi(r^2) r^3 dr}{\int_0^{\infty} \varphi(r^2) r^3 dr}$$

where M is the coefficient to get an integer part of dN_i .

For a microcanonical distribution [3] when number of particles per a cell is not a function of r and

$$\varphi(r^2) = \delta(r^2 - 1) \quad (1.5)$$

one can use the following algorithm to get the initial particle distribution.

The number of particles in the elementary volume is

$$dN_i = M \cdot \left\{ \Delta\theta_1 - \sin(\Delta\theta_1) \cdot \cos(2\theta_{1i} + \Delta\theta_1) \right\} \cdot \sin\left(\theta_{2i} + \frac{\Delta\theta_2}{2}\right) \cdot \sin(\Delta\theta_2)$$

Angle numbers θ_1, θ_2 are defined equal and θ_3 -coordinate of the particle on the surface of the hypersphere is an accidental coordinate.

To provide some symmetry in the particle distribution the particles are generated in the first coordinate quarter and after this the particle coordinates are reflected relatively coordinate axis's.

To get the longitudinal coordinates of the particles to model the compression mechanism in the bunch compressor one can use the following method. As the first step one can generate gaussian distribution of longitudinal coordinates and energy of the particles with given parameters and after this define the longitudinal velocity for each particle using the result of generation in the transverse directions.