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Interdependence of parameters of an X-ray FEL

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Abstract

This paper presents a dimensionless analysis of a self amplified spontaneous emission (SASE) FEL operating in an X-ray wavelength band. Using similarity techniques we have performed an analysis of the results of numerical simulations and derived simple design formulae for calculation of characteristics of the SASE FEL. We have shown also that the growth of the energy spread due to the quantum fluctuations of synchrotron radiation imposes a limit on the minimal achievable value of the wavelength in the X-ray FEL.

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1 Introduction

R&D works on linear colliders give a promise to obtain in the nearest future high-energy, low emittance and monochromatic electron beams which could be used for a wide range of applications. One of the possible applications of these beams is to use them as driving beams for a SASE FEL (self amplified spontaneous emission free electron laser) operating in the VUV and X-ray wavelength band [1–3].

One of the problems of the SASE FEL design consists in calculation of the characteristics of such an FEL amplifier. Some important characteristics could be calculated using the steady-state approach. This approach allows to calculate rather rigorously characteristics of the output radiation (field gain, transverse field distribution and directivity diagram), to estimate the bandwidth of the SASE FEL, the efficiency at saturation and the saturation length.

At the design stage of a SASE FEL usually the problem arises how to choose optimal parameters. As a rule, numerical simulation codes or codes based on fitting formulae are used for optimization of the FEL parameters [2–6]. Nevertheless, the possibility to calculate specific numerical examples, could hardly provide a deep insight into the physics of a SASE FEL and to understand the interdependence of the FEL parameters. Here we should remember that the FEL theory allows to use the similarity technique which not only reduces the number of problem parameters but translates the variables into others possessing a clearer physical sense (see ref. [9] for more detail). Each physical factor influencing the FEL operation (diffraction, space charge, energy spread etc.) is expressed in terms of its own reduced parameter. For each effect the respective reduced parameter indicates its significance for FEL operation. Using the asymptotic behaviour of the solutions, it is possible to derive simple dimensionless relations between the parameters of the problem.

Extending the approach presented in paper [6], in the present paper we have used the similarity techniques for analysis of the results of numerical simulations and derived design formulae for characteristics of an X-ray SASE FEL. In particular, these formulae allow to optimize the field gain, to obtain optimal parameters of the external focusing of the electron beam, etc.

With increasing electron energy the effects connected with the synchrotron radiation of the electrons become to play a significant role. We have shown in this paper that the growth of the energy spread due to quantum fluctuations of synchrotron radiation imposes a limit on the minimal achievable value of the wavelength in X-ray FELs.

2 Basic equations

We consider an FEL amplifier with helical undulator and axisymmetric electron beam [1]. H_{w} and λ_{w} are the amplitude of the magnetic field and the period of the undulator, respectively. The angle of the electron rotation in the undulator is equal to $\theta_{\mathrm{s}} = K/\gamma$, where $\gamma = \mathcal{E}_0/m_{\mathrm{e}}c^2$ is the relativistic factor of the electron with nominal energy $\mathcal{E}_0$, $K = eH_{\mathrm{w}}\lambda_{\mathrm{w}}/2\pi m_{\mathrm{e}}c^2$ is the undulator parameter, $(-e)$ and m_{e} are the charge and the mass of the electron, respectively, and c is the velocity of light (we use CGS units in this paper).

We assume the transverse phase space distribution of the particles in the beam to be Gaussian and the beam is matched to the magnetic focusing system of the undulator. The rms beam size and rms angle spread of the electrons in the beam are given by the expressions:

$$\sigma_{\mathrm{r}} = \sqrt{\epsilon_{\mathrm{n}}\beta/\gamma}, \qquad \sigma_{\theta} = \sqrt{\epsilon_{\mathrm{n}}/\beta\gamma} \quad , \tag{1}$$

where β is the beta function and ϵ_{n} is the rms normalized emittance. We assume the energy spread to be Gaussian:

$$dw = \frac{\exp(-\mathcal{E}^2/2\sigma_{\mathrm{E}}^2)}{\sqrt{2\pi\sigma_{\mathrm{E}}^2}} d\mathcal{E} \quad . \tag{2}$$

Operation of the FEL amplifier can be described in terms of the gain parameter Γ and the following dimensionless parameters: the diffraction parameter B, the space charge parameter $\hat{\Lambda}_{\mathrm{p}}^2$, the parameter of the longitudinal velocity spread $\hat{\Lambda}_{\mathrm{T}}^2$ and the efficiency parameter $\rho_{3\mathrm{D}}$. For the case of an axisymmetric Gaussian beam profile described by eq. (1) these parameters are as follows [6–9]:

$$\begin{aligned}
\Gamma &= \left[I\omega^2\theta_{\mathrm{s}}^2/(I_{\mathrm{A}}c^2\gamma_{\mathrm{z}}^2\gamma)\right]^{1/2} \quad , \\
B &= 2\Gamma\sigma_{\mathrm{r}}^2\omega/c \quad , \\
\hat{\Lambda}_{\mathrm{p}}^2 &= 2c^2(\theta_{\mathrm{s}}\sigma_{\mathrm{r}}\omega)^{-2} \quad , \\
\hat{\Lambda}_{\mathrm{T}}^2 &= \Lambda_{\mathrm{T}}^2/\Gamma^2 = (\sigma_{\mathrm{E}}^2/\mathcal{E}_0^2 + \gamma_{\mathrm{z}}^4\sigma_{\theta}^4/4)/\rho_{3\mathrm{D}}^2 \quad , \\
\rho_{3\mathrm{D}} &= c\gamma_{\mathrm{z}}^2\Gamma/\omega.
\end{aligned} \tag{3}$$

Here $\lambda = 2\pi c/\omega$ is the radiation wavelength, I is the beam current, $I_A = mc^3/e \simeq 17$ kA is Alfven's current and $\gamma_{\mathrm{z}}^2 = \gamma^2/(1+K^2)$.

[1] The case of a planar undulator is considered in Appendix A.

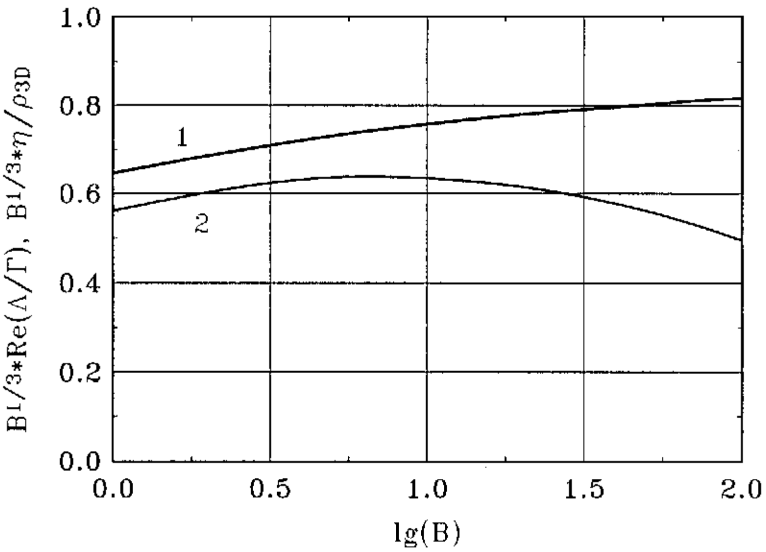


Fig. 1. Normalized field gain $B^{1/3}\,\mathrm{Re}\,\Lambda/\Gamma$ (1) and the FEL efficiency at saturation $B^{1/3}\eta/\rho_{\mathrm{3D}}$ (2) versus the diffraction parameter. Here $\hat{\Lambda}_{\mathrm{p}}^2 = 0$, $\hat{\Lambda}_{\mathrm{T}}^2 = 0$ and detuning corresponds to exact resonance.

In the linear high-gain limit the radiation of the electron beam in the undulator can be represented as a set of modes:

$$E_{\mathrm{x}} + iE_{\mathrm{y}} = \exp[i\omega(z/c - t)] \sum_{n,k} A_{nk}\Phi_{nk}(r)\exp[\Lambda_{nk}z + in\phi] \quad . \tag{4}$$

When amplification takes place, the mode configuration in the transverse plane remains unchanged while the amplitude grows with the undulator length exponentially. Each mode is characterized by the eigenvalue Λ_{nk} and the field distribution eigenfunction $\Phi_{nk}(r)$ in terms of transverse coordinates. The mode with the highest gain (i.e. with the highest value of $\mathrm{Re}\,\Lambda_{nk}$) has the advantage over all other modes. Following the gain process along the undulator axis one finds that the field distribution is governed by the mode with the maximal gain. It was shown in ref. [7] that it is the fundamental TEM_{00} mode that has advantage with respect to all higher modes.

The field gain and the efficiency $\eta_{\mathrm{sat}} = P_{\mathrm{sat}}/P_{\mathrm{beam}}$ of the FEL amplifier as functions of the diffraction parameter are presented in Fig.1 (assuming negligible both space charge and longitudinal velocity spread effects). It can be shown that the value of the normalized field gain $B^{1/3}\,\mathrm{Re}(\Lambda/\Gamma)$ at large values of the diffraction parameter B approaches asymptotically the value $\sqrt{3}/2$ given by the one-dimensional theory (to go over to one-dimensional asymptotic, one should remember that the 1-D gain parameter equals $\Gamma_{\mathrm{1D}} = \Gamma B^{-1/3}$ while the ρ parameter [10] equals $\rho = \rho_{\mathrm{3D}}B^{-1/3}$ [7,8]). On the other hand, the normalized efficiency at saturation $B^{1/3}\eta/\rho_{\mathrm{3D}}$ is far from the value 1.37 given by the 1-D model [11,12]. This is due to the fact that at the linear stage of amplification, in the high gain limit, the fundamental TEM_{00} mode is formed. The nonuniformity of the radial distribution of the fundamental TEM_{00} mode is increased with the value of the diffraction parameter. As a result, diffraction effects play a significant role in the nonlinear stage of the FEL amplifier operation even in the case when formally all the conditions of the 1-D model are fulfilled (see refs. [8,9] for more details).

The process of amplification in the SASE FEL starts from noise and the bandwidth of the output radiation is about

$$\Delta\lambda/\lambda \simeq 2\rho_{3\mathrm{D}} B^{-1/3} \quad , \tag{5}$$

with the resonant wavelength given by the resonance condition:

$$\lambda = \frac{\lambda_{\mathrm{w}}}{2\gamma^2}(1+K^2) \quad . \tag{6}$$

It was shown in papers [13,14] that spikes appear in the output radiation of a SASE FEL. The typical length of each spike is about $l_{\mathrm{sp}} \simeq 2\pi\lambda l_{\mathrm{g}}/\lambda_{\mathrm{w}}$, where l_{g} is the field gain length. The number of spikes during the electron pulse duration $2\sigma_{\mathrm{z}}/c$ is about $N_{\mathrm{sp}} \simeq 2\sigma_{\mathrm{z}}/l_{\mathrm{sp}}$. Relative fluctuations of the output radiation power and fluctuations of the saturation length are about $1/\sqrt{N_{\mathrm{sp}}}$ and $l_{\mathrm{g}}/\sqrt{N_{\mathrm{sp}}}$, respectively.

3 Optimization of parameters of SASE FEL

The analysis of parameters of proposed VUV and X-ray SASE FELs shows that the region of parameters of these devices is at large values of the diffraction parameter, $B > 1$, and negligibly small influence of the space charge, $\hat{\Lambda}_{\mathrm{p}}^2 B^{2/3} \ll 1$. These features allow to perform a simple dimensionless analysis of all important characteristics of FEL amplifiers in the X-ray wavelength band.

First, one can find from Fig.1 that within a wide region of diffraction parameters, $1 \lesssim B \lesssim 100$, the radiation power gain length L_{g} and the efficiency of the FEL amplifier at the saturation η_{sat} could be approximated by the following simple formulae (assuming negligible space charge and longitudinal velocity spread effects):

$$\begin{aligned} L_{\mathrm{g}} &= 1/2\max(\mathrm{Re}(\Lambda)) \simeq 0.7\Gamma^{-1}B^{1/3} \, , \\ \eta_{\mathrm{sat}} &= P_{\mathrm{sat}}/P_{\mathrm{beam}} \simeq 0.6\rho_{3\mathrm{D}}B^{-1/3} \, . \end{aligned} \tag{7}$$

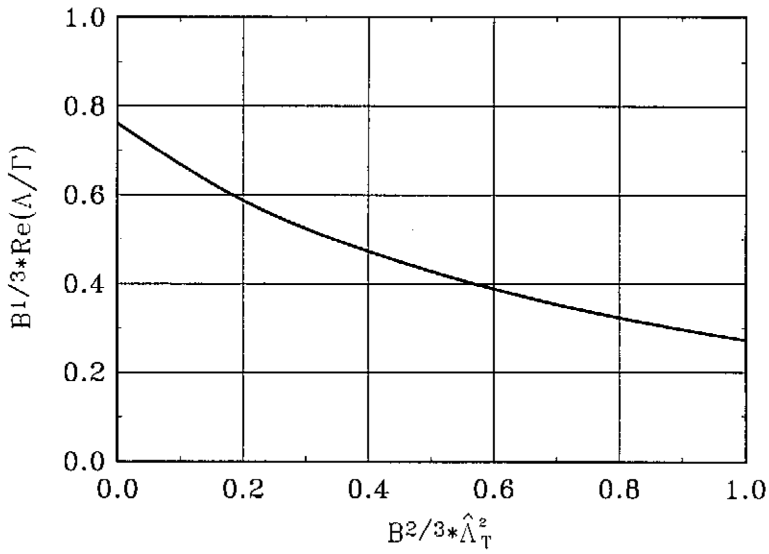


Fig. 2. Normalized field gain $B^{1/3}\,\mathrm{Re}\,\Lambda/\Gamma$ versus the scaled parameter of longitudinal velocity spread $B^{2/3}\hat{\Lambda}_{\mathrm{T}}^2$. Here $B = 10$, $\hat{\Lambda}_{\mathrm{p}}^2 = 0$ and detuning corresponds to the maximum field gain at each value of $\hat{\Lambda}_{\mathrm{T}}^2$.

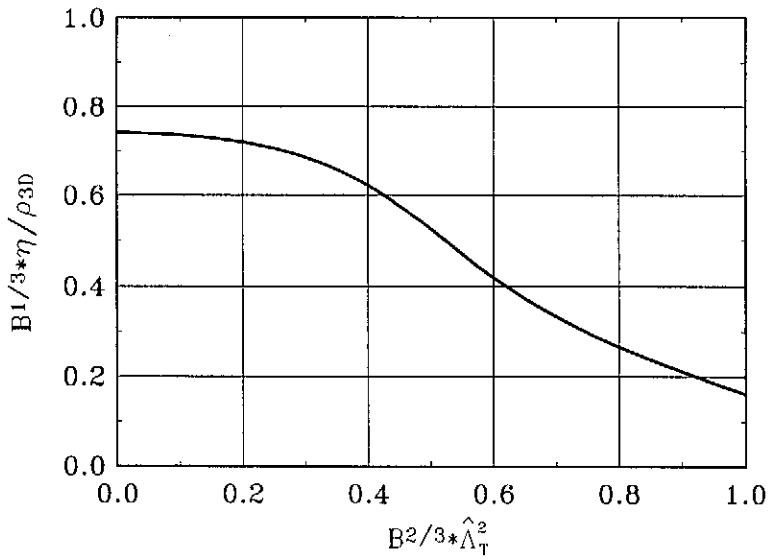


Fig. 3. Normalized FEL efficiency at saturation $B^{1/3}\eta/\rho_{3\mathrm{D}}$ versus the scaled parameter of the longitudinal velocity spread $B^{2/3}\hat{\Lambda}_{\mathrm{T}}^2$. Here $B = 10$, $\hat{\Lambda}_{\mathrm{p}}^2 = 0$ and detuning corresponds to the maximum field gain at each value of $\hat{\Lambda}_{\mathrm{T}}^2$.

The next problem to be solved at a design stage of an experiment is to find a safety margin for the FEL parameters which provide efficient operation. Figs.2 and 3 show the dependencies of the maximal field gain and the efficiency at saturation on the longitudinal velocity spread parameter, respectively. Let us define the safety margin of the FEL amplifier operation by the condition that the relative loss in the field gain due to spread of longitudinal velocities should be less than 10 per cent of the value at $\hat{\Lambda}_{\mathrm{T}}^2 = 0$. This corresponds to the following restriction on the parameter of the longitudinal velocity spread (see Fig.2):

$$\hat{\Lambda}_{\mathrm{T}}^2 B^{2/3} \lesssim 0.1 \quad . \tag{8}$$

Remembering the definition of the longitudinal velocity spread parameter, we can rewrite condition (8) as[2]:

$$\rho_{\mathrm{3D}}^{-2} B^{2/3} \sigma_{\mathrm{E}}^2 / \mathcal{E}_0^2 + (2\pi\epsilon/\lambda)^4 B^{-4/3} \lesssim 0.1 \quad . \tag{9}$$

Analyzing condition (9), we can rewrite it in the following form:

$$\lambda^{4/3} \gtrsim \alpha_1 \beta^{2/3} + \alpha_2 \beta^{-4/3} \quad , \tag{10}$$

where we have shown explicitly the parametric dependency on the radiation wavelength λ and on the focusing beta function β. It follows from this condition and definition of the reduced parameters (3) that at the focusing beta function

$$\beta_{\mathrm{cr}} = \frac{\epsilon\gamma^2}{\sqrt{2}} \frac{\mathcal{E}_0}{\sigma_{\mathrm{E}}} \frac{1}{1+K^2} \quad , \tag{11}$$

we achieve operation of the FEL amplifier at the shortest possible (critical) wavelength which still lies within the safety margin of operation:

$$\lambda_{\mathrm{cr}} \simeq 18\pi\epsilon \frac{\sigma_{\mathrm{E}}}{\mathcal{E}_0} \left[\frac{\gamma I_A}{I} \frac{1+K^2}{K^2} \right]^{1/2} \quad . \tag{12}$$

At a large value of the undulator parameter, $K \gg 1$, the limitation on the minimal wavelength takes the following simple form:

$$\lambda_{\mathrm{cr}} \simeq 18\pi\epsilon \frac{\sigma_{\mathrm{E}}}{\mathcal{E}_0} \left[\frac{\gamma I_A}{I} \right]^{1/2} \quad . \tag{13}$$

[2] Some authors (see, e.g. refs. [15,16]) write the conditions for the FEL amplifier operation as (here we use original notations from refs. [15,16]):
1) $\Delta\mathcal{E}/\mathcal{E} \lesssim \rho$,
2) $\epsilon \lesssim \lambda\!\!\!^{-}/2$.
Remembering that the 1-D parameter ρ is equal to $\rho = \rho_{\mathrm{3D}} B^{-1/3}$, we obtain that the first condition corresponds to the first term in eq. (9), but the second condition considerably overestimates the requirements on the value of the emittance at large values of the diffraction parameter B, as it is seen from eq. (9).

The critical undulator period is defined by the value of the critical wavelength (12) and the resonance condition (6): $\lambda_{\rm w}^{\rm cr} = 2\gamma^2\lambda_{\rm cr}/(1+K^2)$.

When the FEL amplifier operates at the wavelength $\lambda \simeq \lambda_{\rm cr}$ we obtain the following expressions for the power gain length $L_{\rm g}^{\rm cr}$ and the FEL efficiency $\eta_{\rm sat}^{\rm cr}$ (see eq. (7)):

$$L_{\rm g}^{\rm cr} \simeq 1.6\epsilon\gamma^2 \left[\frac{\gamma I_A}{I}\frac{1}{K^2(1+K^2)}\right]^{1/2} \quad ,$$
$$\eta_{\rm sat}^{\rm cr} \simeq 2.3\frac{\sigma_{\rm E}}{\mathcal{E}_0} \quad . \tag{14}$$

In practice there could be a situation when due to technical limitations it is impossible to design a FEL operating at the shortest possible wavelength (for instance, problems of undulator manufacturing or problems to achieve the required value of the optimal beta function). In this case, the operating wavelength has to be chosen to be larger than the minimal one and the problem arises how to optimize this general case. When $\lambda > \lambda_{\rm cr}$, we obtain from eqs. (9) and (10) that the beta function $\beta = \alpha\beta_{\rm cr}$ must be inside the limits:

$$\beta_{\rm min} \lesssim \beta \lesssim \beta_{\rm max} \quad , \tag{15}$$

The tolerable range of factor α is limited by the roots of the equation:

$$2\alpha^2 - 3\left[\frac{\lambda}{\lambda_{\rm cr}}\right]^{4/3}\alpha^{4/3} + 1 = 0 \quad . \tag{16}$$

This dependency is illustrated in Fig.4. In the asymptotic case $\lambda \gtrsim 1.5\lambda_{\rm cr}$, the safety limits for the beta function are given with sufficient accuracy by:

$$\beta_{\rm cr}\left(\frac{1}{3}\right)^{3/4}\frac{\lambda_{\rm cr}}{\lambda} \lesssim \beta \lesssim \beta_{\rm cr}\left(\frac{3}{2}\right)^{3/2}\left[\frac{\lambda}{\lambda_{\rm cr}}\right]^2 . \tag{17}$$

At values of the beta function $\beta \gtrsim \beta_{\rm max}$, the operation of the FEL amplifier is destroyed due to the influence of the energy spread, and at values of the beta function $\beta \lesssim \beta_{\rm min}$ FEL operation is ruined by large longitudinal velocity spread connected with strong external focusing. Within these limits the value of the beta function should be chosen as small as possible in order to increase the field gain and the FEL efficiency, because the field gain increases with the beam current density.

For an FEL amplifier operating within the safety range of the beta function

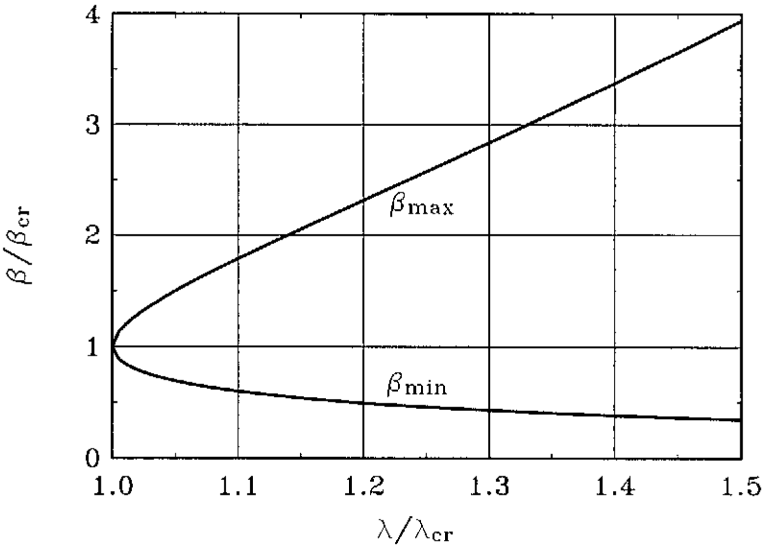


Fig. 4. Safety limits for the external beta function versus the radiation wavelength.

given by eq. (15), we obtain the following expressions for the power gain length L_g and the FEL efficiency η_{sat}:

$$L_g \simeq L_g^{cr} \left[\frac{\lambda}{\lambda_{cr}}\right]^{1/3} \left[\frac{\beta}{\beta_{cr}}\right]^{1/3} ,$$
$$\eta_{sat} \simeq \eta_{sat}^{cr} \left[\frac{\lambda}{\lambda_{cr}}\right]^{2/3} \left[\frac{\beta_{cr}}{\beta}\right]^{1/3} . \qquad (18)$$

In conclusion of this section it should be noted that all formulae obtained above are valid for the case of a large diffraction parameter $B > 1$.

4 Radiation into higher harmonics and quantum fluctuation effects

For $K \gg 1$ there is, on top of the FEL radiation process, also considerable incoherent spontaneous radiation into higher harmonics of the undulator [17]. The mean energy loss of each electron into coherent radiation is given by (for $K \gg 1$):

$$d\mathcal{E}_0/dz = 2r_e^2\gamma^2 H_w^2(z)/3 \quad , \qquad (19)$$

where $r_e = e^2/mc_e^2$. This contribution obviously increases with energy. If the energy loss of the electron $\Delta\mathcal{E}_{\mathrm{SR}}$ is about

$$\frac{\Delta\mathcal{E}_{\mathrm{SR}}}{\mathcal{E}_0} \sim \rho_{\mathrm{3D}} B^{-1/3} \quad , \tag{20}$$

this effect begins to influence the operation of the FEL amplifier. In principle, this does not lead to a fatal limitation of the maximal electron energy, because the energy losses of the electron can be compensated by an appropriate tapering of the magnetic field of the undulator, thus keeping the resonance condition. It limits though the possibility to tune the photon wavelength by tuning the electron energy to the tuning range

$$\frac{|\Delta\lambda|}{\lambda} \lesssim \frac{\eta}{\Delta\mathcal{E}_{\mathrm{SR}}/\mathcal{E}_0} \quad . \tag{21}$$

Also, one might expect considerable experimental difficulties (e.g. heat load on mirrors and monochromators, signal-to-noise ratio, etc), if the totally radiated power is some orders of magnitude larger than the desired FEL power.

A more fundamental limit is imposed by the growth of the uncorrelated energy spread in the electron beam due to the quantum fluctuations of synchrotron radiation. In the following, we treat this for the case of $K \gg 1$. However, in contrast to the effect discussed at the beginning of this section, the quantum noise effect is not expected to disappear for $K \ll 1$. The rate of energy diffusion is given by the expression (for $K \gg 1$):

$$< d(\delta\mathcal{E})^2_{\mathrm{qf}}/dz >= 55 e\hbar\gamma^4 r_e^2 H_{\mathrm{w}}^3/24\sqrt{3} m_e c \quad . \tag{22}$$

This effect is growing drastically with energy. When the induced energy spread becomes comparable with the initial energy spread in the beam σ_{E0}:

$$< (\delta\mathcal{E})^2_{\mathrm{qf}} >^{1/2} \sim \sigma_{\mathrm{E0}} \ , \tag{23}$$

it may dominate the amplification process. This noise effect imposes a principle limit on achieving very short wavelengths. Indeed, to achieve a shorter wavelength at specific parameters of the electron beam (i.e. at specific values of the peak current, the normalized emittance and the energy spread), the energy should be increased (see eq. (12). On the other hand, the gain length is increased drastically with increasing the energy (see eq. (14) which forces to increase the value of the undulator parameter (hence, to increase the undulator field). As a result, at some value of the energy, the energy spread caused by quantum fluctuations will stop the FEL amplifier operation.

To obtain a notion about this limit, let us consider the following model situation. First, consider three parameter sets of the electron beam: $\epsilon_{\mathrm{n}} = 10^{-4}$ cm rad, $\epsilon_{\mathrm{n}} = 2 \times 10^{-4}$ cm rad and $\epsilon_{\mathrm{n}} = 3 \times 10^{-4}$ cm rad. The peak current for all of the examples is equal to $I = 5$ kA and the initial energy spread is equal to $\sigma_{\mathrm{E0}} = 1$ MeV. Imagine that one has a possibility to construct an undulator of length L_{w}. It is reasonable to formulate the problem as follows. At which energy of the electron beam and at which parameters of the undulator the minimal wavelength could be achieved and what is the value of that wavelength.

Fig.5 presents the plots of the minimal achievable wavelength and the corresponding energy of the electron beam versus the undulator length. When performing calculations we assumed that to obtain saturation of a SASE FEL, the undulator length L_{w} should be about 20 power gain lengths L_{g} [6]. The operating wavelength has been obtained using expression (12) with the energy spread given by summing up mean squared values of the initial energy spread σ_{E0} and the energy spread due to the fluctuations of synchrotron radiation (23) at the undulator exit, $\sigma_{\mathrm{E}} = (\sigma_{\mathrm{E0}}^2 + \sigma_{\mathrm{qf}}^2)^{1/2}$. The value of optimal beta function has been calculated in accordance with eq. (11). It is seen from these plots that there is no significant decrease of the minimal wavelength for undulator lengths exceeding $L_{\mathrm{w}} \sim 100$ m.

Fig.6 presents plots of the minimal achievable wavelength and the corresponding energy of the electron beam versus the undulator length. Here we fix the value of the normalized emittance and change the initial energy spread in the beam σ_{E} from zero value up to 6 MeV. It is seen that after $L_{\mathrm{w}} \gtrsim 100$ m all the wavelength curves approach asymptotically the curve describing the case of zero value of the initial energy spread. This indicates that quantum fluctuations of synchrotron radiation impose a limit on the value of the minimal achievable wavelength in an X-ray FEL. The value of this limit can be estimated analytically. Let us consider an electron beam with a very small energy spread. We see from (see eq. (12)) that the critical wavelength is then determined by the induced energy spread due to quantum fluctuations. In the same way as it was done above, we assume that the undulator length L_{w} should be about 20 power gain lengths L_{g}. Using eqs. (12), (14) and (22) one can obtain that at fixed values of the undulator length L_{w}, beam current I and normalized emittance ϵ_n, the absolute minimum of the wavelength is achieved at the undulator parameter $K = 1$. The minimal wavelength and the energy at which this minimum is achieved are given by the following expressions:

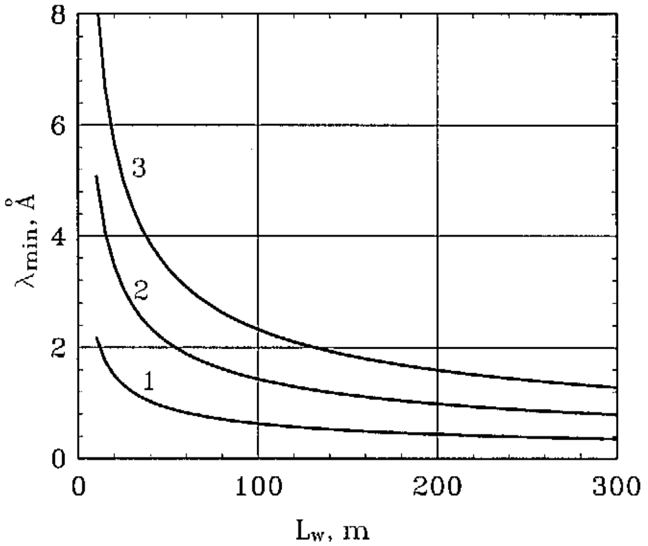
$$\lambda_{\min} \simeq 45\pi \left[\lambda_{\mathrm{c}} r_{\mathrm{e}}\right]^{1/5} L_{\mathrm{w}}^{-7/15} \left[\epsilon_n^2 \frac{I_A}{I}\right]^{8/15} ,$$

$$\gamma \simeq 0.13 \left[\frac{L_{\mathrm{w}}}{\epsilon_n}\right]^{2/3} \left[\frac{I}{I_A}\right]^{1/3} , \qquad (24)$$

where $\lambda\!\!\!^{-}_c = \hbar/mc$. It should be noted that these formulae give only an estimation of the limit, because they have been obtained using expression (22) for the rate of the energy diffusion which is valid only for $K \gg 1$. Fig.7 presents a 3-D view of the critical wavelength versus the length of the undulator and the energy of the electron beam for the optimum case of $\sigma_{E0} = 0$, i.e. the energy spread is determined only by quantum fluctuations.

The next numerical example illustrates the dependence of the critical wavelength on the value of the initial energy spread in the beam σ_{E0} (see Fig.8). Here we have fixed the length of the undulator to $L_w = 100$ m. It is seen form these plots that operation of an X-ray FEL could be possible even at a relatively big value of the initial energy spread in the beam. The price for a bad beam quality is the increased required value of the electron beam energy. To obtain a feeling about optimized parameters of the FEL amplifier, we present in Table 1 three parameter sets corresponding to the plots in Fig.8.

In conclusion of this section we should summarize the following. In principle, quantum fluctuations impose a limit to achieving short wavelengths. The only real possibility to decrease the minimal wavelength is to decrease the value of the normalized emittance. At the present level of accelerator technology it could be possible to construct electron accelerators with a peak current of few kA, a normalized emittance of about 10^{-4} cm rad and an uncorrelated energy spread in the beam about one MeV. At these electron beam parameters the minimal achievable wavelength in an X-ray FEL will be in the range of 0.5 – 1 Å.



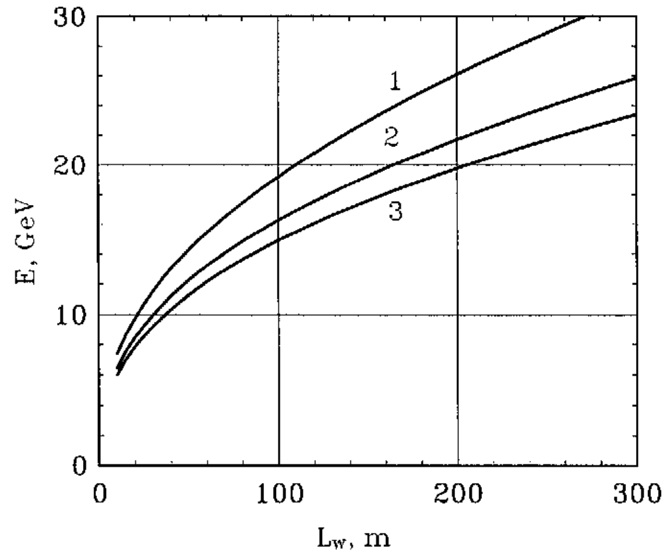
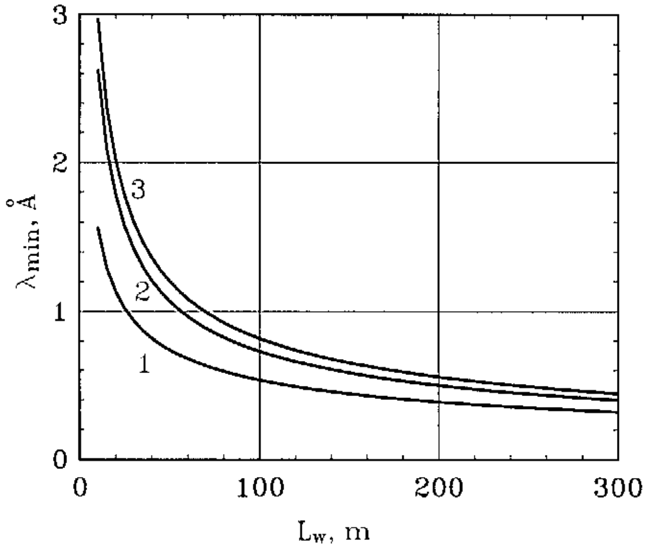


Fig. 5. Minimal achievable photon wavelength in an FEL amplifier and corresponding energy of the electron beam versus the length of the undulator L_w. The curves 1, 2 and 3 correspond to values of the normalized emittance 10^{-4} cm rad, 2×10^{-4} cm rad and 3×10^{-4} cm rad, respectively. The energy spread at the entrance of the undulator in all cases is equal to $\sigma_{E0} = 1$ MeV.



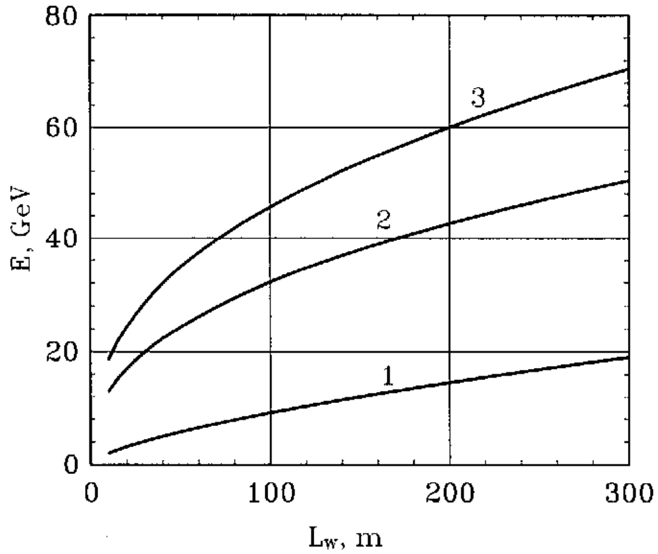


Fig. 6. Minimal achievable photon wavelength in an FEL amplifier and corresponding energy of the electron beam versus the length of the undulator L_w. The curves 1, 2 and 3 correspond to values of the energy spread at the entrance of the undulator of 0 MeV, 3 MeV and 6 MeV, respectively. The normalized emittance is 10^{-4} cm rad in all cases.

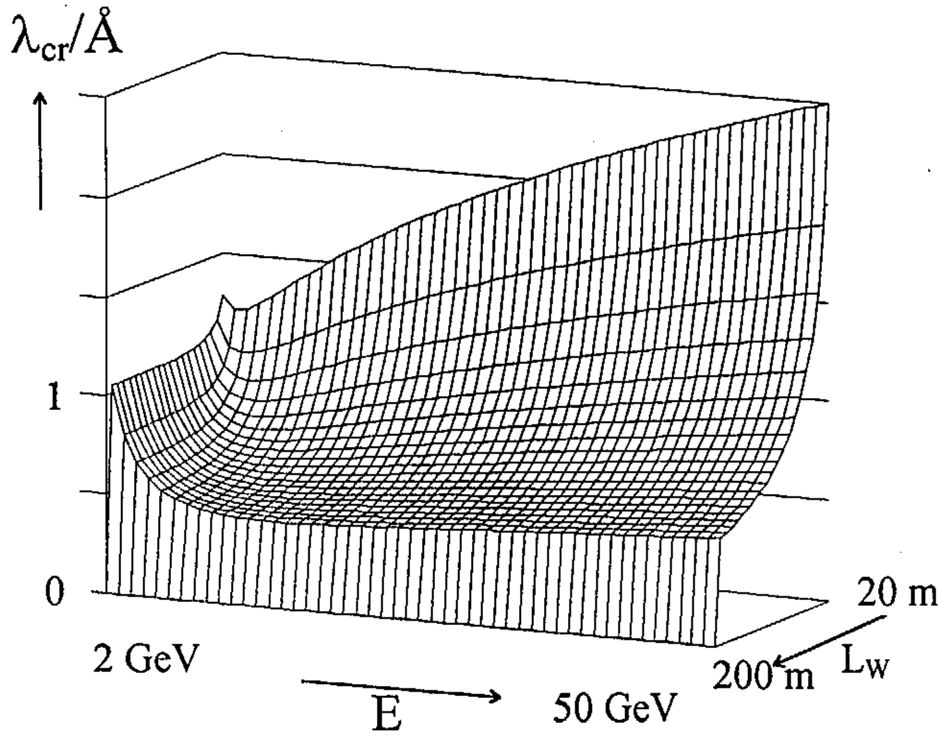
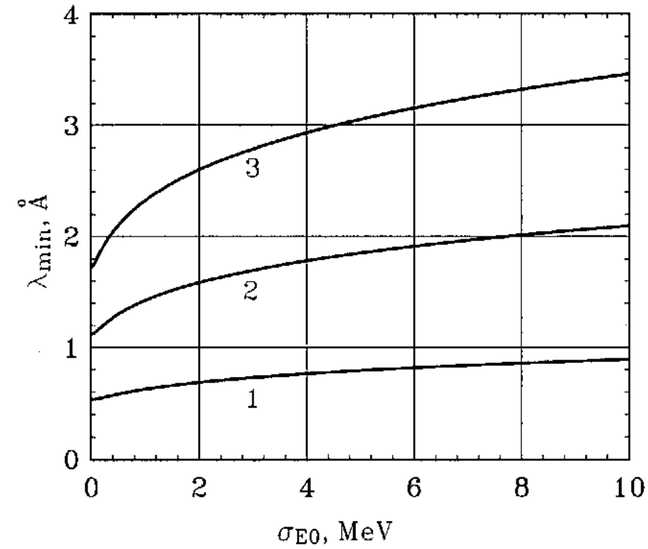


Fig. 7. 3D plot of the critical wavelength as a function of the electron energy (left to right) and undulator length (back to front). Here the normalized emittance is equal to 10^{-4} cm rad, $I = 5$ kA and the initial energy spread is equal to zero.



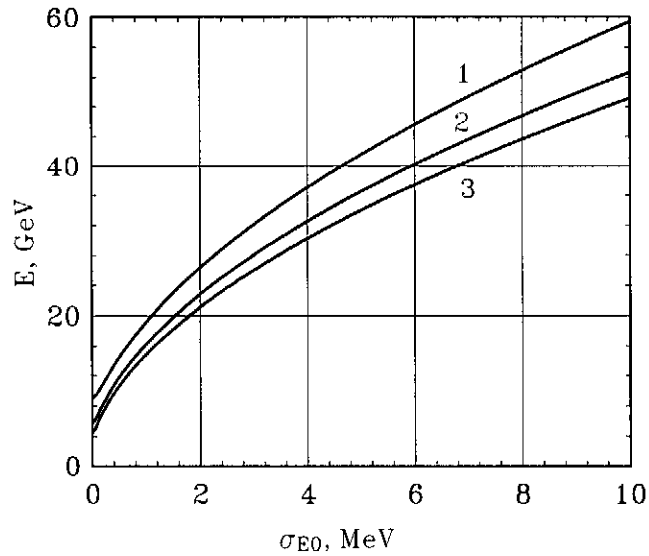


Fig. 8. Minimal wavelength and corresponding energy of the electron beam versus the energy spread at the entrance of the undulator. The curves 1, 2 and 3 correspond to the values of the normalized emittance 10^{-4} cm rad, 2×10^{-4} cm rad and 3×10^{-4} cm rad, respectively. The length of the undulator is equal to 100 m.

Table 1
FEL amplifier for the shortest wavelength

	# 1	# 2	# 3
Electron beam			
Energy $\mathcal{E}_0$, GeV	19.2	22.8	26
Peak current I, kA	5	5	5
RMS normalized emittance ϵ_{n}, cm rad	10^{-4}	2×10^{-4}	3×10^{-4}
RMS energy spread σ_{E}, MeV	1	2	3
External focusing beta function β, m	54	36	27
Undulator*			
Period λ_{w}, cm	3.65	5.42	6.93
Magnetic field H_{w}, T	0.57	0.64	0.69
Undulator parameter K	1.96	3.27	4.46
Undulator length L_{w}, m	100	100	100
Radiation			
Wavelength λ, Å	0.62	1.59	2.76
Power gain length L_{g}, m	5	5	5
Efficiency η, %	0.023	0.035	0.045

* Helical tapered undulator.

5 Conclusion

In this paper, using similarity techniques, we performed an analysis of the interdependence of parameters of the FEL amplifier. Introducing the notion of the safety margin of the FEL amplifier operation (which means such a region of parameters where energy spread and emittance effects could be neglected), we have obtained expressions for the shortest wavelength λ_{cr} which could be amplified at specific parameters of the FEL amplifier (i.e. beam energy, beam current, energy spread, emittance and the undulator parameter). We have also obtained universal formulae describing the main characteristics of the FEL amplifier at this critical point: for the optimal value of beta function, for the gain length and for the saturation efficiency. The operation of the FEL amplifier above the critical point, i.e. at $\lambda \geq \lambda_{cr}$, could be also described by simple design formulae.

We have considered the influence of the synchrotron radiation on the FEL amplifier operation. The first effect is additional energy loss by the electron. This effect imposes a limit on the energy of the beam if using an untapered undulator. The second effect is determined by excitation of energy spread in the beam due to quantum fluctuations of synchrotron radiation. This effect imposes a limit on the way to very short wavelengths.

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Appendix A. Planar undulator

All the results obtained above refer to the case of the helical undulator. Here we present formulae for the case of a planar undulator with amplitude of the magnetic field H_{w}. The undulator parameter K, the angle of electron oscillations θ_{s}, the longitudinal relativistic factor γ_{z} and the factor A_{JJ} are defined as follows:

$$
\begin{aligned}
K &= eH_{\mathrm{w}}\lambda_{\mathrm{w}}/2\pi m_{\mathrm{e}}c^2 \ , \\
\theta_{\mathrm{s}} &= K/\gamma \ , \\
\gamma_{\mathrm{z}}^2 &= \gamma^2/(1+K^2/2) \ , \\
A_{JJ} &= J_0(K^2/(4+2K^2)) - J_1(K^2/(4+2K^2)) \ ,
\end{aligned}
\qquad (\mathrm{A.1})
$$

where J_0 and J_1 are the Bessel functions.

The gain parameter Γ, the diffraction parameter B, the space charge parameter $\hat{\Lambda}_{\mathrm{p}}^2$, the parameter of the longitudinal velocity spread $\hat{\Lambda}_{\mathrm{T}}^2$ and the efficiency parameter ρ_{3D} are defined as:

$$
\begin{aligned}
\Gamma &= \left[IA_{JJ}^2\omega^2\theta_{\mathrm{s}}^2/(2I_{\mathrm{A}}c^2\gamma_{\mathrm{z}}^2\gamma)\right]^{1/2} \ , \\
B &= 2\Gamma\sigma_{\mathrm{r}}^2\omega/c \ , \\
\hat{\Lambda}_{\mathrm{p}}^2 &= 4c^2(\theta_{\mathrm{s}}\sigma_{\mathrm{r}}\omega A_{JJ}^2)^{-2} \ , \\
\hat{\Lambda}_{\mathrm{T}}^2 &= \Lambda_{\mathrm{T}}^2/\Gamma^2 = \omega^2(\sigma_{\mathrm{E}}^2/\mathcal{E}_0^2 + \gamma_{\mathrm{z}}^4\sigma_\theta^4/4)/(c^2\gamma_{\mathrm{z}}^4\Gamma^2) \ , \\
\rho_{\mathrm{3D}} &= c\gamma_{\mathrm{z}}^2\Gamma/\omega.
\end{aligned}
\qquad (\mathrm{A.2})
$$

The parameters of the FEL amplifier at the critical point are as follows:

$$
\begin{aligned}
\lambda_{\mathrm{cr}} &\simeq 18\pi\epsilon\frac{\sigma_{\mathrm{E}}}{\mathcal{E}_0}\left[\frac{2\gamma I_A}{IA_{JJ}^2}\frac{1+K^2/2}{K^2}\right]^{1/2} \ . \\
\beta_{\mathrm{cr}} &\simeq \frac{\epsilon\gamma^2}{\sqrt{2}}\frac{\mathcal{E}_0}{\sigma_{\mathrm{E}}}\frac{1}{1+K^2/2} \ , \\
L_{\mathrm{g}}^{\mathrm{cr}} &\simeq 1.5\epsilon\gamma^{5/2}\left[\frac{2I_A}{IA_{JJ}^2}\frac{1}{K^2(1+K^2/2)}\right]^{1/2} \ , \\
\eta_{\mathrm{sat}}^{\mathrm{cr}} &\simeq 2.3\frac{\sigma_{\mathrm{E}}}{\mathcal{E}_0} \ .
\end{aligned}
\qquad (\mathrm{A.3})
$$

For the case of operating wavelength $\lambda > \lambda_{\mathrm{cr}}$ formulae (18) are applicable.

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