Microbunch Amplification in the European XFEL

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1 Introduction

Linac-based X-ray-free-electron lasers (XFELs) require very short bunches of highbrightness electron beams with peak currents of the order of kilo-Amperes. These bunches cannot be produced directly in guns because space charge forces would destroy the brilliance within a short distance [1]. Increasing bunch peak currents with longitudinal compression at beam energies high enough to suppress space charge forces sufficiently to avoid emittance increase cannot be achieved alone by inducing velocity differences along the bunch. Instead or in addition, dispersive sections like magnet chicanes where particle path length depends on energy have to be employed. An energy chirp induced to the bunch with an rf system upstream of the chicane will then result in longitudinal bunch compression and peak current increase.

Bunch compression is usually done in stages. On one hand the energy has to be high enough to avoid too strong self effects, not only by space charge forces but also by coherent radiation and compression work. A too strong compression in the first stage leads to tight tolerances, and a too high energy level of that stage increases the effort for a higher harmonic system needed to linearize and compensate rf curvature and non linear dispersion. Too high energy in further stages is limited by magnet strengths, magnet dimensions, chicane dimensions, remaining energy chirp and finally by effects due to incoherent radiation.

In total, the bunch compression system consists of a sequence of straight acceleration sections, where initially strong space charge forces are reduced by the energy gain, and chicanes which increase the peak current, which restores the space charge forces roughly to their initial strength. Then the bunch enters the next stage and the process is repeated.

It was shown that this staged bunch compression with its interplay of straight sections where space charge forces are present and magnet chicanes where path length depends on electron energy can very effective in amplifying small initial intensity modulations of the bunch [2]. The gain of that amplification at certain wavelengths can be strong enough to generate strong beam intensity modulations at the end of the bunch compression system even if the initial modulation is just provided by shot noise.

This mechanism has been baptized 'micro-bunching instability'. It effects, depending on the degree of intensity modulation, several aspects of machine operation: the direct impact on the lasing process is presently discussed and calculated, strong modulation will cause emittance degradation due to CSR fields in the magnet chicanes and even small modulations in the optical wavelength regime render OTR screens which are used to measure transverse beam properties, useless. The LCLS project has been suffering from the latter.

The same paper which predicts the instability also offers a mitigation: uncorrelated energy offsets of electrons lead to longitudinal position mixing in the magnet chicanes and damps the instability. A so-called 'laser heater', where the electron bunch traverses an undulator magnet together with a laser beam, increases the uncorrelated energy offsets of the electrons and suppresses the instability strength to a tolerable level.

In this paper, we present methods to calculate micro-bunching amplification gains and results for the case of the European XFEL.

1.1 The European XFEL

The European XFEL uses a super-conducting L-band linac to accelerate beams with peak currents of about 5 kA to an energy of up to 20 GeV. The beam then passes up to 200 m long undulator magnets, where the SASE process (Self-Amplification of Spontaneous Emission) produces coherent X-rays with wavelengths down to 1 Å. The injector incorporates a photoemission radio-frequency (rf) gun that produced bunches of about 1 nC charge, 50 A current and 5 MeV energy. Space charge forces in the bunch vary with the radius and longitudinal position. The radial blowup of the bunch is counteracted by a focusing scheme (known as emittance compensation [3, 4]) that uses a magnetic solenoid and an accelerating section as shown in Fig. 1. Elaborate numerical studies using the code ASTRA have been carried out to optimize the injector geometry, coil arrangement and amplitudes and phases of rf fields. After the first accelerating module, at a beam energy of 130 MeV, the slice energy spread is increased by a socalled, laser heater' (see Fig. 2) from about 1 keV to up to 30 keV. The shape of a typical laser heater spectrum with 10 keV rms is compared in Fig. 3 with a gaussian distribution. The required peak current of ≈ 5 kA is achieved in a multi stage bunch compression system by a longitudinal compression at 500 MeV and 2 GeV, see Fig. 4. The longitudinal compression by a factor of about twenty in the first stage and totally of about hundred increases the induced slice energy spread by the same factor to 3 MeV. This is of the same order as the spread caused by incoherent synchrotron radiation in the undulator. Simulation calculations have shown that slice energy spread bigger than that starts reducing SASE intensity. This limits the amount of damping of the micro-bunching instability the laser heater can provide and requires the optimization of the bunch compression system with respect to overall micro-bunching gain.

1.2 Organization of the Report

Compression of the longitudinal distance between two particles can be caused due to differences of velocities and path lengths. Both depend on energy. In this report we distinguish a regime where velocity effects are not negligible or even might play a main role (section 2) from the ultra relativistic regime where the approximation v = c is used (section 3). A typical parameter characterising velocity effects is the wavelength L_p of plasma oscillations. The constant velocity theory is applicable if L_p is large compared to longitudinal linac dimensions or the length to double the energy.

In one dimensional theory the plasma wavelength can be estimated as

$$L_p \approx 2\pi c \left(\frac{e}{m_e} \frac{\omega |Z'(\omega)|}{\gamma^3} \frac{I}{c}\right)^{-1/2}$$

with I the beam current, γ the relativistic factor, ω the angular frequency of micro modulation and $Z'(\omega)$ the longitudinal space charge impedance per length.(Z' is estimated in section 3.2.1.) For parameters typical for the European XFEL the plasma wavelength is plotted in Fig. 6 as function of modulation frequency. The relativistic factor of 15 and 250 corresponds approximately to the energy before and after the first accelerating module. It is obvious that the constant velocity theory is applicable after that module for frequencies below 10 THz. It will be shown that micro-bunch amplification will be strongly suppressed in beams with initial rms energy spread of 10 keV for larger frequencies.

The low energy part considered in section 2 extends from the gun to the end of the first accelerating module. Especially effects in the gun are dominated by strong space charge forces so that simple perturbation theory fails to estimate the evolution of the bunch shape even without micro modulations. Therefore the tracking code ASTRA is used to calculate all effects. The approach of ASTRA is described and methods to solve the space charge problem are discussed. Detailed convergence studies have been done for a bench mark example based on plasma oscillations and for all computations of the injector geometry.

For the constant velocity part a linear theory is summarised, extended and applied in section 3. It is assumed that amplitude and wavelength of the initial modulation are small. The small amplitude is required for the linearization of self effects while a small wavelength allows to neglect non linearities caused by external fields (rf curvature and non linear dispersion). A coasting beam with a superimposed harmonic modulation is used to calculate a frequency dependent gain function. Longitudinal impedances are estimated and compared (section 3.2.1). It is shown that space charge effects followed by coherently radiated fields are most essential. If impedances in dispersive sections are negligible the problem can be reduced to a discrete system which can be solved fast and efficient (sections 3.2.2 and 3.2.3.b). A longitudinal impedance depending on frequency and position in the linac but offset independent is used for dispersive trajectories leading to a one dimensional integral equation (section 3.2.3) or a reduced tracking approach (section 3.2.4).

The application uses the space charge model for linear sections (diagnostics and accelerators) and/or CSR impedances in dispersive sections (bunch compressors). Both

impedance types are approximated in steady state, the CSR impedance without SC contributions. It is demonstrated that the linear gain exceeds easily values of 10^5 for beams with small energy spread, while longitudinal heating is efficient to reduce the gain below 10^2 . The influences of SC and CSR impedances are compared and it is found that good estimations can be achieved without CSR effects in dispersive sections.

Section 4 generalizes the initial conditions of the linear theory for constant velocity and uses the numerical results from the low energy part to compute the total gain in the multi stage system of the European XFEL.

1.3 Notation

Longitudinal coordinates along the linac (or beam line coordinates) are written in uppercase letters as S, A, B. Section 3.2.2 considers particle properties only in discrete planes as S_0 , S_1 , S_a , S_b . To simplify the notation of phase space vectors and transport matrices we replace in this section the beam line coordinates by their indices that are either integer numbers or lower case letters. Sections 3.2.3 and 3.2.4 use continuous beam line coordinates that might be indexed if they are related to a certain reference plane.

Each particle is represented by phase space coordinates

$$\mathbf{X} = \left(\begin{array}{cccc} x & x' & y & y' & z & \eta\end{array}\right)^t$$

with x, x', y, y' the horizontal and vertical trace space coordinates, z is the longitudinal coordinate into the direction of motion and $\eta = (\mathcal{E} - \mathcal{E}_{ref})/\mathcal{E}_{ref}$ is the relative energy deviation from the reference energy \mathcal{E}_{ref} . The linear transformation from reference plane A to B is

$$\mathbf{X}_B = \mathbf{Q}_{B \leftarrow A} \mathbf{X}_A$$

with the matrix elements

$$\left(\mathbf{Q}_{B\leftarrow A}\right)_{ij} = q_{B\leftarrow A}^{(ij)}$$

For a set of particles the particle index is written after the position index separated by a comma; f.i. $X_{B,n}$ or $z_{B,n}$ for the longitudinal coordinate.

The concept of wake fields and impedances is used to calculate the interaction of particles [20]. The wake function describes the effect of a source particle (in unperturbed motion) to a test particle while the wake potential considers a smooth source distribution. In principle the longitudinal charge density is

$$\Lambda_A(z) = q \sum \delta(z - z_{A,i}) ,$$

with q the charge of one particle or macro-particle and the summation over all particles of the bunch. Usually the charge density is treated as smooth function supposing that an appropriate smoothing method has been applied that keeps micro and macro structures but removes shot noise. The calculation of a one dimensional wake potential as the longitudinal $W_{||}(z)$ assumes further a certain transverse distribution function and neglects that the (longitudinal) field depends on the transverse position of the test particle. With these restrictions the energy change of a particle between reference plane A and B is

$$\eta_B \mathcal{E}_{\mathrm{ref},B} = \eta_A \mathcal{E}_{\mathrm{ref},A} + e W_{||}(z)$$
 .

Here we assumed a longitudinally rigid beam $(z_B = z_A = z)$. The energy change per length (beam line position) is

$$\frac{d\eta}{dS}\mathcal{E}_{\rm ref} = eW'_{||}(z) = eE_{||}$$

with $W'_{||}(z)$ the longitudinal wake potential per length that coincides with the longitudinal electrical field $E_{||}$. The impedance is the frequency domain description of the wake function. Usually an ultra relativistic beam is supposed (with $k = \omega/c$ and z = -t/c). Therefore the longitudinal wake potential is related to the longitudinal impedance $Z(\omega)$ or $Z'(\omega)$ by:

$$W_{||}(z) = -\frac{1}{2\pi} \int Z(\omega)I(\omega) \exp(-jkz)d\omega \text{ with } I(\omega) = \int \Lambda(z) \exp(jkz)dz$$

or

$$W'_{||}(z) = -\frac{1}{2\pi} \int Z'(\omega) I(\omega) \exp(-jkz) d\omega$$

The impedance per length Z' and the modulation gain G are considered in the following as function of different parameter sets. The most general form of Z' is written in curly brackets, for instance $Z'\{\omega, S, \gamma(S), \sigma(S), ...\}$, with the first parameter the angular frequency and the other parameters as beam or geometry properties that may be related to the beam line coordinate S. The impedance per length is a local quantity and therefore ω has also 'local' meaning. The short notation in round brackets $Z(\omega)$ is used for integrated impedances while

$$Z'[S] = Z'\{\omega_0 \times C(S), S, \gamma(S), \sigma(S), \dots\}$$

pronounces the dependency on the beamline coordinate and the suppressed angular frequency is initial frequency ω_0 at the entrance S_0 times the compression factor C(S). The gain function $G\{k_0, S_0 \to S\}$ is the modulation gain from the initial wavenumber k_0 at S_0 to the local position S. It may be written as function of the beam line coordinate $G[S] = G\{k_0, S_0 \to S\}$ supposing a certain initial wavelength or as function of 'compressed' frequency

$$\tilde{G}(\omega) = G\{\omega/(cC(S)), S_0 \to S\}$$

supposing a certain beam line position.

2 Low Energy Model

Here, the low energy regime is considered to be the first 14.5 m of the European XFEL wherein the bunch is accelerated to an energy of 130 MeV. This part belongs to the injector and includes the rf gun (Figure 7) and the first acceleration section [6]. The rf gun cavity is a normal conducting 1.3 GHz cavity of one and a half cells with a high accelerating field of 60 MV/m at the cathode. A solenoid with a maximum field of 0.2 T is centered at 0.4 m downstream of the cathode. After a drift of about 3 m follows the first accelerating section with two times four cavities at ≈ 3.3 m. The rf field has a maximal amplitude of 23 MV/m for the first four cavities and 34 MV/m for the next four cavities, respectively. The fields of the solenoid and the cavities are plotted in Figure 8 and 9.

The bunch charge is given with 1 nC which results in a current of approximately 50 A. For the simulations, the laser profile is modeled as a flat top shape with a 20 ps full width at half maximum (FWHM) and a rise and fall time of 2 ps. The transverse profile of the bunch is assumed to be radially uniform with a radius $r_0 = 1.5$ mm. In principle, the rotational symmetry is kept until the bunch exits the first accelerator section.

2.1 Tracking Code ASTRA

The simulations for microbunch amplifications at low energies were performed with the tracking code ASTRA (<u>A space charge tracking algorithm</u>) [7]. The program package ASTRA has been successfully used in the design of linac and rf photoinjector systems. The ASTRA suite originally developed by K. Flöttmann tracks macro particles through user defined external fields including the space charge field of the particle cloud.

The first version of ASTRA allowed for the calculation of space charge fields of bunches with azimuthal symmetry only. A further development was the implementation of a FFT-based Poisson solver for full 3D space charge calculations with free space boundary conditions [8]. Recently, a new set of 3D Poisson solvers has been implemented in ASTRA by G. Pöplau. These Poisson solvers are an improved FFT Poisson solver based on the integrated Green's function and iterative algorithms, among them the state-of-the-art multigrid Poisson solver.

The space-charge calculations are performed within the tracking procedure, where the trajectories of N macro particles are computed by means of the relativistic equations of motion given by [9]

$$\frac{\mathrm{d}\gamma_{i}\boldsymbol{v}_{i}}{\mathrm{d}t} = \frac{q}{m}(\boldsymbol{E}_{i} + \boldsymbol{v}_{i} \times \boldsymbol{B}_{i}),
\frac{\mathrm{d}\boldsymbol{x}_{i}}{\mathrm{d}t} = \boldsymbol{v}_{i} = \frac{\gamma_{i}\boldsymbol{v}_{i}}{\sqrt{\gamma_{i}\boldsymbol{v}_{i}^{2}/c^{2} + 1}}, \qquad i = 1, \dots, N.$$
(1)

Here, \boldsymbol{x}_i and \boldsymbol{v}_i are the position and the velocity of the macro particle *i*, while *q* and *m* are the charge and the mass of an elementary particle, where the macro particles represent the distribution of all particles in a bunch. Further, $\gamma_i := (1 - \boldsymbol{v}_i^2/c^2)^{-1/2}$ denotes the Lorentz factor and *c* the speed of light. The electric field \boldsymbol{E}_i and the

magnetic flux B_i are the superposition of external and self-induced fields (the so-called space charge forces) at the position of the *i*-th macro particle. Instantly with a change in position of the particles the space charge field changes as well. The field has to be recomputed after certain time steps of the numerical integration of the relativistic equations of motion (1).

2.2 2D Space Charge Model

The 2D space charge model assumes that the 3D particle distribution of the bunch is cylindrically symmetric. Hence, the space charge calculations can be reduced to formulas in 2D. The bunch is longitudinally discretized into N_{long} equidistant slices and transversally into N_{rad} rings. The radial grid height can be varied by the factor *Cell_var* between one and two. For our simulations we chose *Cell_var* = 2, which means that the inner ring has the double height of the outer ring. The static field of the bunch is calculated in the average rest frame. The calculation is performed by numerically integrating over the rings, where a constant charge density is assumed. More details can be found in the ASTRA manual [7].

2.3 3D Space Charge Model

A widely used method for the calculation of 3D space charge fields is the particle mesh method (PM) described e. g. in [10]. The space charge fields are calculated again in the average rest frame of the bunch. For the PM approach generally, a rectangular box, in the following denoted as Ω , is constructed around the bunch. Then, a Cartesian grid is defined inside the box and the values of the space charge density ρ are assigned at the grid points by a volume-weighted distribution of the charge of the macro particles. Let ε_0 denote the dielectric constant and G the Green's function given by

$$G(x, y, z) = \frac{1}{r(x, y, z)}$$
 with $r = r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

The potential ϕ can be calculated either by the convolution with the Green's function

$$\phi(x,y,z) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathbb{R}^3} G(x-x',y-y',z-z')\varrho(x',y',z') \,\mathrm{d}x' \,\mathrm{d}y' \,\mathrm{d}z' \tag{2}$$

or by Poisson's equation

$$-\Delta\phi(x,y,z) = \frac{\varrho(x,y,z)}{\varepsilon_0} \quad \text{in } \Omega \subset \mathbb{R}^3.$$
(3)

The application of a Poisson solver provides the potential ϕ at the mesh points.

In ASTRA two different solver types are available. One of these methods is the widespread FFT Poisson solver which uses formulation (2). With the approach (2) the potential of the bunch decays asymptotically with r^{-1} in free space. The FFT Poisson solver itself provides a periodic solution but the potential has the correct behavior in the region of interest (see [10] and subsection 2.3.1).

The second approach is a fast iterative solver based on multigrid. It is applied on Poisson's equation (3) together with the boundary conditions

$$\frac{\phi(x, y, z)}{\partial n} = 0 \quad \text{on } \partial\Omega_1,$$

$$\frac{\partial\phi(x, y, z)}{\partial n} + \frac{1}{r_b(x, y, z)}\phi(x, y, z) = 0 \quad \text{on } \partial\Omega_2,$$
(4)

where $r_b(x, y, z)$ denotes the distance between the center of the bunch and the boundary. On the surface $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ ($\partial \Omega_1 \cap \partial \Omega_2 = \emptyset$) perfectly conducting boundaries ($\partial \Omega_1$) or open (free space) boundaries ($\partial \Omega_2$) can be applied. The second equation of (4) approximates free space boundary conditions with an asymptotic decay of r^{-1} . This approximation can be applied properly if the computational domain is large enough. For a long bunch (according to the transformation into the rest frame) as it occurs in the first accelerating section the enlargement of Ω would be so big that this approach is no longer efficient.

2.3.1 FFT Poisson Solver

The FFT Poisson solver is based on the integral approach (2), which is discretized with step sizes h_x , h_y , h_z in x-, y- and z-direction, respectively. It yields

$$\phi_{i,j,k} = \frac{h_x h_y h_z}{4\pi\varepsilon_0} \sum_{i',j',k'} G_{i-i',j-j',k-k'} \cdot \varrho_{i',j',k'},\tag{5}$$

where $\phi_{i,j,k}$, $G_{i-i',j-j',k-k'}$ and $\rho_{i',j',k'}$ refer to the values of the related functions at the mesh points. Applying the Discrete Fourier Transformation (DFT) the relation

$$\hat{\varphi}_{l,m,n} = \hat{G}_{l,m,n} \hat{\varrho}_{l,m,n} \tag{6}$$

is obtained from (5) due to the convolution theorem. The circumflex denotes the DFT and (l, m, n) the harmonic wave numbers. The inverse DFT provides the potential at the grid points. It is well-known that the DFT can be efficiently calculated by Fast Fourier Transformation (FFT) algorithms.

It is shown in [11, 12] that the FFT Poisson solver together with the standard Green's function does not calculate the fields correctly if the bunch is either very long or very short. Thus, the idea of the integrated Green's function \tilde{G} is introduced in [13] with

$$\tilde{G}(x,y,z) = \int_{x_{i'}-h_x/2}^{x_{i'}+h_x/2} \mathrm{d}x' \int_{y_{j'}-h_y/2}^{y_{j'}+h_y/2} \mathrm{d}y' \int_{z_{k'}-h_z/2}^{z_{k'}+h_z/2} \mathrm{d}z' G(x-x',y-y',z-z').$$

The analytical formula for the integral is given in [11] with

$$\begin{aligned} \iiint \frac{1}{r(x,y,z)} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \\ &- \frac{z^2}{2} \arctan \frac{xy}{zr(x,y,z)} - \frac{y^2}{2} \arctan \frac{xz}{yr(x,y,z)} - \frac{x^2}{2} \arctan \frac{yz}{xr(x,y,z)} \\ &+ yz \ln(x + r(x,y,z)) + xz \ln(y + r(x,y,z)) + xy \ln(z + r(x,y,z)). \end{aligned}$$

Only recently this solution method was implemented in ASTRA and successfully applied [14, 15].

The mesh for the FFT Poisson solver is equidistant and the number of steps has to be a power of 2. More precisely, the bunch is longitudinally discretized into $N_{long} = N_z - 2N_{z0} - 1$ steps, where $N_z = 2^t$ and N_{z0} is a small number of steps that is used to model some additional space around the bunch (default $N_{z0} = 2$). Thus, the bunch is discretized for instance by $N_{long} = 251$ mesh cells if $N_z = 256$. The same concept is used for the discretization in transverse direction, i. e. the bunch is discretized into $N_x - 2N_{x0} - 2$ and $N_y - 2N_{y0} - 2$ steps, respectively. The Poisson solver itself is performed on a grid with $2N_x \times 2N_y \times 2N_z$ mesh points, but no extra space is required as for the FD approach (see subsection 2.3.2) if the bunch becomes long in the rest frame. The space charge density is set to zero at the additional grid points and the (integrated) Green's function is extended periodically due to [10, 11].

2.3.2 Iterative Poisson Solvers

Iterative Poisson solvers require a different approach. Firstly, the Laplacian in (3) is discretized. The discretization by second order finite differences (FD) with simultaneous consideration of the boundary conditions (4) provides a linear system of equations

$$\mathbf{A}u = f,\tag{7}$$

where u denotes the vector of the unknown values of the potential and f the vector of the given space charge density at the grid points. Since the matrix **A** is sparse, iterative solvers can be applied efficiently.

During the simulations it turned out that the numerical noise of the calculated field was quite high for bunch energies above 50 MeV. The FFT space charge routine provided less numerical noise and could be applied with much less effort than the iterative solvers. Thus, only the FFT Poisson solver with the integrated Green's function was used as 3D space charge model for the simulations.

2.4 Plasma Oscillations

For charged bunches at low energy the periodic oscillation between a current density modulation and an energy modulation is referred to as **plasma oscillation**. Hereby space charge forces transform a current density modulation into an energy modulation. On the other side an energy modulation is transformed into current modulation due to large relative velocity spread in the low energy regime [16].

A self consistent, analytical model for longitudinal plasma oscillation in a relativistic electron beam is given by Geloni et al. in [17]. Furthermore, these authors benchmark the tracking code ASTRA with the analytical results in [18], where the 2D space charge routine is used. We applied the experimental setup given in [18] in order to confirm the results for the plasma oscillation and further to benchmark the 3D space charge routine of ASTRA with the results obtained in [18].

Following the numerical experiment in [18] we considered a cylindrically shaped bunch with uniform particle distribution. The transversal dimension was $r_0 = 1.0$ mm and the bunch length $l_b = 2.2$ cm. The bunch propagated with kinetic energy $E_{kin} = 6.0$ MeV. The total charge of the bunch was set to Q = 3.3 nC such that the current yielded I = 45 A. The particle distribution was computed with the program generator [7] that generated the corresponding input distribution for ASTRA.

Then, the current was modulated with an amplitude of $\rho_m = 0.05 = 5\%$ and a wavelength $\lambda = 1$ mm. The modulation was generated by shifting the z-components of the macro particles due to

$$z = z_0 - m\sin(\frac{2\pi}{\lambda}z_0) \tag{8}$$

with the modulation index $m = \frac{\lambda \rho_m}{2\pi}$. Whereas the simulations in [18] are performed with $12 \cdot 10^6$ macro particles we obtained comparable results with $1 \cdot 10^6$ macro particles.

Furthermore, the bunch was focused by means of a solenoidal field $B_z = 7$ T in order to suppress the transverse evolution of the bunch. The bunch was tracked along a distance of 7 m with a maximal time integration step of 10 ps. The tracking distance corresponds to the characteristic plasma oscillation length $\lambda_p = 7.31$ m given in [18].

The space charge calculations were performed both with the 2D and the 3D model of ASTRA. The parameters for the 2D model were set due to [18]: we used 10 radial rings with $Cell_var = 2$. The bunch length was discretized with N_{long} steps. In [18] it is shown that at least 20 steps per wavelength are necessary in order to obtain acceptable results. Figures 10 and 11 show the evolution of current and energy modulation along the tracking distance of 7 m for the different numbers $N_{long} = 100, 200, 400, 800$. They confirm the results of Geloni et al.: whereas a discretization with 100 and 200 steps was not sufficient, the results obtained with 400 and 800 steps coincide very well.

The 3D space charge calculations were performed with $N_x = N_y = 32$, $N_{x0} = N_{y0} = N_{z0} = 2$ and $N_z = 64, 128, 256, 512$, respectively. Figures 12 and 13 represent the results for plasma oscillation calculated with the 3D space charge method. They show very good agreement with the 2D space charge routine. As for the 2D case it turns out that at least 20 steps per wavelength are necessary in order to achieve the theoretical result.

2.5 Simulations for the first 14.5 m of the European XFEL

The simulations for the low energy part were started at the cathode and finished after the first accelerator section according to the set up described above. Thus, the bunch was tracked over the first 14.5 m of the European XFEL. There the bunch has achieved an energy of 130 MeV.

In order to simulate the space charge oscillation dynamics in the low energy region ASTRA was used together with full 3D space charge calculations. It turned out that the application of the 2D model of ASTRA is also possible because the bunch maintains the cylindrical symmetry until the end of the first accelerating section. A convenient side effect was that the simulation times with the 3D FFT solver were less than 50% of the simulation times with 2D space charge calculations considering the same number of macro particles. Since 3D space charge calculations were not yet implemented for field calculations at the cathode, the simulations were started at the cathode with 2D

λ [mm]	N	N_z	$\sharp \text{ steps}/\lambda$
1.20	5,000,000	256	35
1.00	5,000,000	256	29
0.75	5,000,000	512	44
0.50	5,000,000	512	29
0.35	5,000,000	1024	41
0.20	10,000,000	1024	23

Table 1: Number of macro particles N, number of longitudinal steps N_z and resulting number of steps per wavelength for different wavelengths.

space charge calculations. Thus, the unmodulated bunch was tracked to the position of 0.07 m, where the mirror charges for field calculations at the cathode were switched off. Here, the bunch had a length of ca. 8.6 mm. The particle distribution at 0.07 m was now modulated by shifting the z-position of the particles due to equation (8) with $\rho_m = 5\%$. Then the tracking was proceeded with the modulated particle distribution applying the 3D space charge model.

For the 3D space charge calculations the bunch was transversally discretized into $N_x = N_y = 32$ steps with $N_{x0} = N_{y0} = 2$. Longitudinally the discretization had to be adapted to the wavelength of the modulation because at least 20 steps per wavelength were needed for the simulations (see subsection 2.4). That means that the number of particles had to be increased with decreasing wavelength. The number of macro particles N and the number of longitudinal steps N_z with the corresponding number of steps per wavelength for the simulations are given in Table 1. The particle distribution was investigated at the following positions: 0.07 m (after the cathode, start of the simulation with modulated particle distribution), 2.0 m (drift), 3.0 m (drift, entrance to the accelerating section), 8.8 m (location after the first four cavities), 14.5 m (exit of the first accelerating section). Assuming the current and energy modulation of the form

$$I(z) = I_0 + \hat{I}\cos(\frac{2\pi}{\lambda}z - \varphi), \tag{9}$$

$$E(z) = E_{ref} + \hat{E}\cos(\frac{2\pi}{\lambda}z - \psi)$$
(10)

the related amplitude and phase shift were calculated by Fourier analysis. Hereby only the core of the bunch was considered in order to avoid edge effects from head and tail. Figures 14–17 demonstrate the influence of the longitudinal resolution for $\lambda =$ 1.0 mm and $\lambda = 0.75$ mm. Thus, the result given for plasma oscillation (section 2.4) that at least 20 steps per wavelength are required is confirmed. The Figures 18 and 19 summarize the simulation results for the wavelength from $\lambda = 1.2$ mm down to $\lambda = 0.2$ mm. They show the amplitude of the modulation of relative current and the modulation of energy, respectively. A change in phase is represented in a change in sign of the related amplitude. Such a shift in phase can be observed for wavelengths equal to or smaller than $\lambda = 0.35$ mm for the energy modulation. Figure 20 represents the results at the location of 14.5 m for the different wavelength. It shows the linear dependency of current and energy modulation with respect to the wavelength. The quantity $\Delta \varphi = \varphi - \psi$ is the difference in phase between current and energy modulation. Here, it can be observed that a phase shift occurs with $\lambda = 0.75$ mm which is caused by the phase shift in the current modulation.

3 Velocity Independent High Energy Model

3.1 Gain Mechanism

The amplification of longitudinal density fluctuation in a charged particle beam is related to the coupling of density to energy and vice versa, see Fig. 21 from [5]. The mechanism of energy modulation is caused by longitudinal self fields that are characterised by longitudinal wakes $W_{\parallel}(z)$, or in frequency domain by the longitudinal impedance $Z(\omega)$. The coupling from energy to the relative position z of a particle in a bunch is known as longitudinal dispersion and characterised by the corresponding element of the transfer matrix $q^{(56)}$. In the velocity independent high energy regime dispersion is caused only by the energy dependency of trajectories and therefore by different path lengths between reference planes. Dispersion and energy chirp are created by purpose in the European FEL to compress bunches. The undesired fluctuation of the chirp due to self fields leads to a fluctuation of the compression factor and might superimpose a current modulation that is larger than the compressed initial fluctuation.

In multi stage compression systems the density and energy variations might be amplified by many orders of magnitude until saturation and beyond. A counter measure is to increase the energy spread of particles with the same longitudinal position. Such particles, that have been initially in the same slice, will be smeared in longitudinal direction and level micro modulations to some extend.

We distinguish between systems where the modifications of energy and longitudinal density is decoupled from systems where they happen simultaneously. The first type can be treated as discrete system, the latter needs continuous phase space manipulations. Multi stage bunch compression systems are of the first type if self fields in the compression chicanes are neglected. Bunch compressors with self fields belong to the second type.

3.2 Linear Gain Model

3.2.1 Impedances

In frequency domain an axial beam current is described by the density

$$\mathbf{J}(\mathbf{r}_{\perp} + z\mathbf{e}_z) = I \,\mathbf{e}_z \psi(\mathbf{r}_{\perp}) \exp(j(\omega - z/\beta c)) ,$$

with $\psi(\mathbf{r}_{\perp})$ the transverse density and I the total current. The longitudinal electrical field in a cylindrical structure is

$$\mathbf{E}(\mathbf{r}_{\perp} + z\mathbf{e}_z) \cdot \mathbf{e}_z = E_z(\mathbf{r}_{\perp}) \exp(j(\omega - z/\beta c))$$
.

For $\omega r/c\gamma \ll 1$ with r a typical transverse beam dimension, the variation of the longitudinal field in the beam is nearly negligible and the longitudinal impedance per length can be defined either by

$$Z' = -I^{-1} \int E_z(\mathbf{r}_\perp) \psi(\mathbf{r}_\perp) d\mathbf{r}_\perp \ ,$$

or by

 $Z' = -I^{-1} E_z(0)$.

The concept of a steady state impedance per length is generalized in c) for circular motion. Geometric impedances are usually calculated per structure, cavity or module but it is useful (for comparisons) to express them per length as it has been done in d).

a) Space charge impedance in free space

The longitudinal space charge impedance is deduced from Maxwells equations for free space. Using Amperes and Faradays laws, the wave equations are satisfied by the electromagnetic potentials:

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} -\varepsilon^{-1}\rho \\ v\mu\rho\mathbf{e}_z \end{pmatrix} , \qquad (11)$$

where ρ is the charge density and $v = \beta c$ a constant longitudinal velocity along the z coordinate. Both equations yield:

$$\mathbf{A} = v\varepsilon\mu\phi\mathbf{e}_z = \frac{\beta}{c}\phi\mathbf{e}_z \quad , \tag{12}$$

with $\beta = vc^{-1}$. For symmetry of revolution the potential ϕ and the space charge density are represented by the following relations:

$$\begin{pmatrix} \rho(\mathbf{r},t)\\ \phi(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} f(r)\\ h(r) \end{pmatrix} \exp(j\omega(t-z/\beta c)) .$$
(13)

Adopting cylindrical coordinates the homogeneous part of (11) becomes w.r.t. (12) and (13):

$$\left(r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} - \left(\frac{rk}{\gamma\beta}\right)^2\right) h(r) = 0 \quad . \tag{14}$$

Here γ represents the Lorentz factor and $k = \omega c^{-1}$. The general solution of this modified Bessel differential equation is given by the modified Bessel function of the first $I_0(\xi r)$ and second $K_0(\xi r)$ kind, with $\xi r = \frac{kr}{\beta\gamma}$. (In later chapters the constants I_{ν} have the meaning of dc currents) Upon using the electromagnetic potentials the z-component of the electric field yields:

$$E_z = \frac{k}{j\gamma^2\beta} \left\{ \begin{array}{c} I_0(\xi r) \\ K_0(\xi r) \end{array} \right\} \exp(j\omega(t - z/\beta c)) , \qquad (15)$$

the corresponding magnetic field yields:

$$H_{\varphi} = -\frac{\beta}{Z_0} \frac{k}{\gamma^2 \beta} \left\{ \begin{array}{c} I_1(\xi r) \\ -K_1(\xi r) \end{array} \right\} \exp(j\omega(t - z/\beta c)) , \qquad (16)$$

as a general solution. The parameter Z_0 means the free space impedance ($Z_0 = \sqrt{\mu_0/\varepsilon_0}$). According to the boundary condition of the electromagnetic field the computation domain of Eq. (15) and (16) is separated as follows:

$$E_{z} = \frac{k}{j\gamma^{2}\beta} \exp(j\omega(t - z/\beta c)) \begin{cases} I_{0}(\xi r)/I_{0}(\xi R) & r < R \\ K_{0}(\xi r)/K_{0}(\xi R) & r > R \end{cases},$$
(17)

and

$$H_{\varphi} = -\frac{\beta}{Z_0} \frac{k}{\gamma^2 \beta} \exp(j\omega(t - z/\beta c)) \begin{cases} I_1(\xi r)/I_1(\xi R) & r < R \\ -K_1(\xi r)/K_1(\xi R) & r > R \end{cases},$$
(18)

where R is the beam radius. Upon using Ampere's law the current of the beam yields:

$$2\pi R \left(H \left(R^+ \right) - H \left(R^- \right) \right) = I \quad . \tag{19}$$

A hollow straw beam satisfies this assumption. Using (19) and (18) the space charge impedance per length in free space is:

$$Z'(\omega) = -j\frac{Z_0}{4\pi}\frac{\omega}{c\gamma^2}T\left(\frac{|\omega|r}{c\gamma}\right) \quad \text{with} \quad \gamma \gg 1 \quad , \tag{20}$$

where T(x) represents a function that depends on the transverse beam profile and x the characteristic transverse beam dimension. For a hollow beam with finite radius r_h but infinitely small thickness the shown straw beam derivation yields

$$r = r_h$$
, $T(x) = T_h(x) = \frac{2/x}{K_1(x)/K_0(x) + I_1(x)/I_0(x)}$

According to the free space formulation of the hollow straw beam a beam with uniform cross section is given in [21]. A pencil beam with the radius r_p yields:

$$r = r_p, \quad T(x) = T_p(x) = \frac{4}{x^2} \left(1 - xK_1(x)\right)$$

Furthermore an impedance of a bunch with a gaussian cross section is given in [22]. Using the rms beam size σ_r , the shape function yields:

$$r = \sigma_r$$
, $T(x) = T_g(x) = \int_{x^2}^{\infty} \frac{\exp(x^2 - t)}{t} dt$.

For low frequencies $|x| \ll 1$ the shape functions become:

$$\begin{split} T_h \left(x \right) &\approx 0.232 - 2 \ln x \quad \approx -2 \ln x / 1.123 \ , \\ T_p \left(x \right) &\approx 1.232 - 2 \ln x \quad \approx -2 \ln x / 1.852 \ , \\ T_g \left(x \right) &\approx -0.577 - 2 \ln x \approx -2 \ln x / 0.749 \ , \end{split}$$

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the shape dependent functions T(x) and their approximations are plotted in Fig. 22. The longitudinal space charge impedance is purely imaginary, this indicates a redistribution of the beam energy without energy losses. We use the real function $X'(\omega, r, \gamma) = jZ'(\omega)$ to characterize it:

$$X'(\omega) = \frac{Z_0}{4\pi} \frac{\omega}{c\gamma^2} T\left(\frac{|\omega|r}{c\gamma}\right) \quad . \tag{21}$$

b) Space charge impedance in beam pipes

Assuming that the beam with radius $r = r_h$ is moving inside of a smooth cylindrical beam pipe with the radius $r = r_p$ the impedance is investigated as a superposition of two hollow free space beams. The current of the inner beam is assumed as I. In order to account metallic resistive wall conductivities a surface impedance boundary condition is used. The current density of a hollow beam with current I and radius r_h is:

$$\mathbf{J}(\mathbf{r}, I, r_h) = I \exp\left(j\omega\left(t - z/\beta c\right)\right) \cdot \mathbf{e}_z \frac{\delta\left(r - r_h\right)}{2\pi r} \; .$$

The electromagnetic field yields:

$$\mathbf{E}(\mathbf{r}, I, r_h) = I \exp\left(j\omega\left(t - z/\beta c\right)\right) \cdot \left(E_r\left(r, r_h\right) \mathbf{e}_r + E_z\left(r, r_h\right) \mathbf{e}_z\right) ,$$

$$\mathbf{H}(\mathbf{r}, I, r_h) = I \exp\left(j\omega\left(t - z/\beta c\right)\right) \cdot H_{\varphi}\left(r, r_h\right) \mathbf{e}_{\varphi} ,$$

with

$$E_{r}(r) = \begin{cases} a \cdot I_{1}(k_{r}r) & 0 \leq r < r_{h}, \\ b_{1} \cdot I_{1}(k_{r}r) + b_{2} \cdot K_{1}(k_{r}r) & r_{h} < r < r_{p}, \end{cases}$$

$$(r, r_{t}) = \frac{1}{2} \begin{cases} a \cdot I_{0}(k_{r}r) & 0 \leq r < r_{h}, \end{cases}$$
(22)

$$E_{z}(r, r_{h}) = \frac{1}{j\gamma} \begin{cases} a \cdot I_{0}(k_{r}r) & 0 \leq r < r_{h}, \\ b_{1} \cdot I_{0}(k_{r}r) - b_{2} \cdot K_{0}(k_{r}r) & r_{h} \leq r < r_{p}, \end{cases}$$
(22)
$$U_{-}(r, r_{h}) = \frac{\beta}{j\gamma} \begin{cases} a \cdot I_{1}(k_{r}r) & 0 \leq r < r_{h}, \\ 0 \leq r < r_{h}, \end{cases}$$
(22)

$$H_{\varphi}(r, r_h) = \frac{\beta}{Z_0} \begin{cases} a \cdot I_1(k_r r) & 0 \le r < r_h, \\ b_1 \cdot I_1(k_r r) - b_2 \cdot K_1(k_r r) & r_h \le r < r_p, \end{cases}$$
(23)

with $\beta = \sqrt{1 - \gamma^{-2}}$ and $k_r = \frac{\omega}{c\gamma\beta}$. At $r = r_h$ the electromagnetic field yields in analogy to (19)

$$a \cdot I_1(k_r r_h) = b_1 \cdot I_1(k_r r_h) + b_2 \cdot K_1(k_r r_h) \quad \text{w.r.t.} \quad (22) ,$$

$$\frac{I}{2\pi r_h} = b_1 \cdot I_1(k_r r_h) - b_2 \cdot K_1(k_r r_h) \quad \text{w.r.t.} \quad (23) .$$

The boundary condition at the beam pipe represents the metallic surface which exhibits an impedance Z_s with resistive as well as inductive components. It is known as surface impedance and it is nearly independent on the curvature of the beam pipe

$$Z_s(\omega) = \sqrt{j\omega\mu/\kappa}$$
,

with the conductivity κ of the pipe material. Note that the conductivity can be frequency dependent. Adopting this impedance at $r = r_p$ the boundary condition for the electromagnetic field at the pipe wall yields:

$$E_z = -Z_s \cdot H_{\varphi}$$

This boundary condition yields at $r = r_p$ w.r.t. the former investigated electromagnetic fields

$$b_1(\frac{1}{j\gamma}I_0(k_rr_p) + Z_s\frac{\beta}{Z_0}I_1(k_rr_p)) + b_2(-\frac{1}{j\gamma}K_0(k_rr_p) + Z_s\frac{\beta}{Z_0}K_1(k_rr_p)) = 0$$

Finally the beam impedance $Z' = -E_z(r_h)/I$ follows as:

$$Z' = -\frac{Z_0}{j\gamma\beta 2\pi r_h} \frac{I_0(k_r r_h)}{I_1(k_r r_h)K_0(k_r r_h) - K_1(k_r r_h)I_0(k_r r_h)} \left(K_0(k_r r_h) + I_0(k_r r_h) \frac{j/\gamma K_0(k_r r_p) + Z_s/Z_0 K_1(k_r r_p)}{j/\gamma I_0(k_r r_p) + Z_s/Z_0 I_1(k_r r_p)} \right)$$

In the case of infinite wall conductivity $\kappa \to \infty$, the beam impedance is obtained by the substitution $Z_s = 0$ and the tangential component of the electric field disappears at $r = r_p$.

c) Impedance from coherent synchrotron radiation

The one-dimensional approach of Borland [23] is widely used to investigate CSR effects in bunch compression systems. It neglects transverse beam dimensions and calculates the longitudinal self-field of a one-dimensional beam that is obtained by a projection of the 'real' three-dimensional beam to a reference trajectory. As the field of a one-dimensional beam is infinite on its trajectory a 'renormalized' Coulomb term is used [24]. Therefore so called space charge effects are not taken into account. (It has to be pointed out that a proper mathematical distinction between SC and CSR effects is not possible. Usually the term SC is associated to effects of a beam in uniform motion and CSR to propagating waves. The renormalization just extracts a field singularity that would be in the same way present if the one dimensional beam would be in linear motion.) The generalized one dimensional approach calculates the longitudinal field of an arbitrary (but frozen) bunch $\Lambda(z)$ by convolution with a kernel function [24, 25]:

$$E_{||}^{\text{renormalized}}(z,S) = \frac{1}{4\pi\epsilon} \int_{-\infty}^{0} \Lambda'(z+u)\tilde{K}(S,u)du \,.$$
(24)

In frequency domain the steady state impedance of a bunch on an arc with radius R_0 and large energy $(\gamma \ll (R_0 \omega/c)^{1/3})$ is

$$Z'_{\rm CSR}(\omega \ge 0) = AZ_0 \sqrt[3]{\frac{\omega}{jcR_{\rm curv}^2}}, Z'_{\rm CSR}(\omega < 0) = \overline{Z'_{\rm CSR}(-\omega)}$$
(25)

with

$$A = \frac{1}{2\pi} \frac{\Gamma(2/3)}{\sqrt[3]{3}} \approx 0.15$$

[26, 27]. In Figs. 23 and 24 the transient CSR impedance (according to [25]) is compared with the steady state limit for an sector magnet with 0.5 m length and $R_0 = 10$ m. In the considered frequency range the steady state impedance can be taken only as rough estimate.

d) Cavity impedance

The longitudinal and transverse wake of bunches with gaussian longitudinal profile in TESLA cavities and modules is investigated in [28]. It is found that in an infinite chain of cavities and modules after some length (depending on the bunch length) a steady state condition is reached. For this condition a pseudo wake function (for point sources) was convoluted and fitted to numerically calculated wakes of bunches with σ down to 50 μ m. The longitudinal pseudo wake of a module with the active length of $L_{\rm a} = 8.28$ m is

$$w_{\text{module}}(z) = W_m \exp(-\sqrt{z/z_m}) \times \begin{cases} 0 & \text{if } s < 0\\ 1 & \text{otherwise} \end{cases}$$

with $W_m = 344 \cdot 10^{12} \text{ V/C}$ and $z_m = 1.74 \text{ mm}$. Therefore the impedance per active length is

$$Z'_{\rm acc}(\omega) = -\int \frac{w_{\rm module}(z)}{cL_a} \exp(j\omega z/c)dz \quad .$$
⁽²⁶⁾

The Fourier transformation was calculated numerically although it can be solved analytically. The result is compared an discussed in the following.

e) Some impedances in the European XFEL

Some impedances **per length** are compared in Fig. 25 for typical beam parameters in the European XFEL and energies of 130 MeV, 500 MeV, 2 GeV and 17.5 GeV. The cavity impedance is nearly energy independent so that the same curve appears on all sub-diagrams. For low frequencies the impedance is essentially caused by cavities and other sources of 'geometric' wakes. In the frequency range of interest the free space impedance increases and is usually larger. The frequencies for an initial modulation wavelength of 0.1 and 1 mm are marked. Due to the compression at 500 MeV with C = 20 and at 2 MeV with C = 5 the window is shifted to higher frequencies. Although the space charge impedance drops with γ^{-2} it is in the marked window the major contribution to the total impedance. The influence of PEC boundary conditions and finite conductivity (copper) are also shown. (The radius of the beam pipe corresponds to the radius of TESLA cavities of 38 mm for sub-diagrams (a) to (c) und to the radius of the beam distribution system of about 20 mm in (d).) Only at 17.5 GeV the influence of metallic conductivity seems to be relevant. Important for micro bunching effects is the **integrated** impedance between dispersive sections as it is calculated in Fig. 26 for the 'high energy part' before BC1, the linac between BC1 and BC2 and the linac from BC2 to the collimator. For these cases the integrated space charge impedance is large compared to the cavity impedance. The effect of higher harmonic cavities is not considered. The integrated effect of resistive wall conductivity is small compared to the integrated free space impedance and is also neglected.

3.2.2 Discrete Approach: Only Space Charge Effects

a) Single stage, no uncorrelated energy spread

The charge density $\Lambda_0(z_0)$ and relative energy deviation $\eta_0(z_0)$ at the entrance S_0 as function of the initial longitudinal position z_0 are

$$\Lambda_0(z_0) = \Lambda_0 \cdot (1 + m\cos(kz_0))$$

$$\eta_0(z_0) = \frac{\mathcal{E}'_0}{\mathcal{E}_0} z_0$$

with $\Lambda_0 = I_0/c$ the dc part, *m* the modulation index, $\lambda = 2\pi/k$ the modulation wavelength, \mathcal{E}_0 the reference energy and \mathcal{E}'_0 the slope of the linear energy chirp. The bunch is accelerated by an rf system that increases the reference energy and chirp to \mathcal{E}_1 and \mathcal{E}'_1 . In the linear from S_0 to the entrance of the bunch compressor S_1 a longitudinal space charge impedance

$$X_{1\leftarrow 0}(\omega) = \int_{S_0}^{S_1} X'_f\{\omega, r(S), \gamma(S)\} dS$$

is experienced due to the longitudinal field

$$E_{||}(S, z_0) = mI_0 X'_f \{kc, r(S), \gamma(S)\} \sin(kz_0)$$

at the position S with $\gamma(S)$ the local relativistic factor. The impedance calculation assumes a round beam with local beam radius r(S) that is interpolated from the horizontal and vertical beam dimensions given by the emittances and the beta functions. The longitudinal coordinate and relative energy deviation at the entrance of the bunch compressor are

$$z_1(z_0) = z_0$$

$$\eta_1(z_0) = \frac{\mathcal{E}'_1}{\mathcal{E}_1} z_0 + \frac{m I_0 X_{1 \leftarrow 0}(kc)}{\mathcal{E}_1/e} \sin(kz_0) .$$

The bunch compressor is characterized by the transport matrix element $q_{2\leftarrow 1}^{(56)}$. Therefore the longitudinal phase space coordinates after the bunch compressor are

$$z_2(z_0) = z_0 + q_{2\leftarrow 1}^{(56)} \eta_1(z_0)$$

$$\eta_2(z_0) = \eta_1(z_0) .$$

/ \

Due to the energy modulation the compression factor depends on the relative longitudinal position:

$$\tilde{C}(z_2(z_0)) = \left(\frac{dz_2}{dz_0}\right)^{-1} = \left(1 + q_{2\leftarrow 1}^{(56)} \frac{\mathcal{E}'_1}{\mathcal{E}_1} + q_{2\leftarrow 1}^{(56)} \frac{mI_0 X_{1\leftarrow 0}(kc)k}{\mathcal{E}_1/e} \cos(kz_0)\right)^{-1}$$

We assume that the modulation index m is sufficiently small and neglect nonlinear terms. Therefore the unmodulated and modulated compression factors are

$$C = (1 + q_{2 \leftarrow 1}^{(56)} \mathcal{E}'_1 / \mathcal{E}_1)^{-1} ,$$

$$\tilde{C}(z_2) = C - C \frac{m I_0 X_{1 \leftarrow 0}(kc) k}{\mathcal{E}_1 / e} \cos(k_2 z_2) + \mathcal{O}(m^2) ,$$

with $k_2 = Ck$. The longitudinal charge density after the compressor $\Lambda_2(z_2(z_0))$ is the product of the initial density $\Lambda_0(z_0)$ and the modulated compression factor $\tilde{C}(z_2(z_0))$:

$$\Lambda_2(z_2) = C\Lambda_0 \left(1 + m \left(1 - q_{2 \leftarrow 1}^{(56)} k_2 \frac{I_0 X_{1 \leftarrow 0}(kc)}{\mathcal{E}_1/e} \right) \cdot \cos(k_2 z_2) \right) + \mathcal{O}(m^2) \; .$$

It can be written in the form

$$\Lambda_2(z_2) = \Lambda_2 \left(1 + mG \cdot \cos(k_2 z_2) \right) + \mathcal{O}(m^2) , \qquad (27)$$

with the unmodulated line charge density $\Lambda_2 = C\Lambda_0$ and the modulation gain

$$G = 1 - q_{2\leftarrow 1}^{(56)} k_2 \frac{I_0 X_{1\leftarrow 0}(kc)}{\mathcal{E}_1/e} .$$
(28)

The gain factor may be considered as function $G\{k, S_0 \to S_2\}$ of the initial wave number k, the position S_0 of initial modulation and the position S_2 of amplified modulation.

Note that this derivation assumes purely imaginary impedances. In four magnet bunch compression chicanes as they are foreseen for the European XFEL $q_{2\leftarrow1}^{(56)}$ is a positive quantity. In the frequency range under consideration ($\omega\sigma_r/c\gamma \ll 1$) $X_{1\leftarrow0}$ is also positive so that the second term takes a negative value. Therefore single stage gain curves to large absolute values start necessarily with a zero crossing at low frequencies followed by a high gain regime with 180 deg phase shift that reaches its extremum when $kX_{1\leftarrow0}(kc)$ gets maximal.

b) Single stage with uncorrelated energy spread

We consider a longitudinal and modulated charge density $\Lambda_0(z_0)$ as before but an uncorrelated energy spread is now superimposed to the systematic chirp. Therefore the beam is characterized by the longitudinal phase space density $\psi_0(z_0, \eta_0)$ with

$$\psi_0(z_0,\eta_0) = \Lambda_0(z_0) F_\eta \left(\eta_0 - \frac{\mathcal{E}_0'}{\mathcal{E}_0} z_0\right) \quad .$$

 $F_{\eta}(\eta)$ with $\int F_{\eta}(\eta) d\eta = 1$ describes the energy spread of longitudinal slices. The compression smears each slice longitudinally. This can be expressed by a convolution

of the smearing function $hF_{\eta}(hz_2)$ and the compressed longitudinal charge density $\Lambda_2(z_2)$:

$$\Lambda_2^{(u)}(z_2) = \int \Lambda_2(z_2 - u) h F_\eta(hu) du ,$$

with $h = \mathcal{E}_0/\mathcal{E}_1 q_{2\leftarrow 1}^{(56)}$ and $\Lambda_2(z_2)$ calculated by Eq. (27). For simplicity we assume a symmetric energy distribution $F_\eta(\eta) = F_\eta(|\eta|)$ and write the density analogously to Eq. (27) as

$$\Lambda_2^{(u)} = \Lambda_2 \left(1 + mG^{(u)} \cos(k_2 z_2) \right) + \mathcal{O}(m^2)$$

with the modified gain

$$G^{(\mathbf{u})} = G \int \cos\left(k_2 q_{2\leftarrow 1}^{(56)} \frac{\mathcal{E}_0}{\mathcal{E}_1} \eta\right) F_\eta(\eta) d\eta \quad .$$
⁽²⁹⁾

F.i. the gain function $G^{(u)}\{k, S_0 \to S_2\}$ for a Gaussian energy distribution with the rms spread $\sigma_\eta \mathcal{E}_0$ is

$$G^{(u)}\{k, S_0 \to S_2\} = G\{k, S_0 \to S_2\} \exp\left(-\frac{1}{2}\left(Ckq_{2\leftarrow 1}^{(56)}\frac{\mathcal{E}_0}{\mathcal{E}_1}\sigma_\eta\right)^2\right)$$
(30)

For large absolute values this result agrees with the estimation in [2]. In the following we do not explicitly distinguish gain factors with or without uncorrelated energy spread and therefore suppress the upper index '(u)'.

Fig. 27 shows the gain curves in the European XFEL due to pure space charge effects upstream of the first bunch compressor. For initial distributions with uncorrelated rms energy spread of 10 keV the maximal gain can be kept below 20. A feature of all gain curves of laser heated initial distributions is $\operatorname{sinc}(x)$ like behavior that is also found in the Fourier spectrum of the density function in Fig. 3.

c) Multi stages, no uncorrelated energy spread

The schematic setup of a multi stage bunch compression system and the used notation are shown in Fig. 28. Without uncorrelated energy spread $(F_{\eta}(\eta) = \delta(\eta))$ the particle energy is a unique function of the relative bunch coordinate. At reference plane *a* (with beam path coordinate S_a) the line charge density and the relative energy offset are:

$$\begin{aligned} \Lambda_a(z_a) &= \Lambda_a \Re \left\{ 1 + mG_a \exp(-jk_a z_a) \right\} \\ \eta_a(z_a) &= \Re \left\{ \frac{\mathcal{E}'_a}{\mathcal{E}_a} z_a - jmH_a \exp(-jk_a z_a) \right\} . \end{aligned}$$

This reference plane is either the entrance of the complete multi stage system (a = 0) or it coincides with the exit of a previous bunch compression chicane or dispersive section. $G_a = G_a\{k, S_0 \to S_a\}$ is the gain of amplitude modulation and H_a stands for

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energy modulation. The scheme starts with $G_0 = 1$ and $H_0 = 0$. The stage ' $a \rightarrow c$ ' includes an rf system and impedance between plane a and b = a + 1. The range of the dispersive compressor is between b and c = a + 2. The transformation from plane 'a' to plane 'b' is characterized by the phase space manipulation

$$\begin{pmatrix} z_b \\ \eta_b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{E}'_{b \leftarrow a}}{\mathcal{E}_b} & \frac{\mathcal{E}_a}{\mathcal{E}_b} \end{pmatrix} \begin{pmatrix} z_a \\ \eta_a \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Re \left\{ m \frac{G_a I_a Z_{b \leftarrow a}(k_a c)}{\mathcal{E}_b/e} \exp(-jk_a z_a) \right\} .$$
(31)

The compression transformation is

$$\left(\begin{array}{c} z_c \\ \eta_c \end{array}\right) = \left(\begin{array}{c} 1 & q_{c \leftarrow b}^{(56)} \\ 0 & 1 \end{array}\right) \left(\begin{array}{c} z_b \\ \eta_b \end{array}\right)$$

As the longitudinal and transverse phase space are decoupled we consider only the longitudinal sub-matrix (matrix elements $q^{(55)}$, $q^{(56)}$, $q^{(66)}$ and $q^{(66)}$). The transport matrix of stage ' $c \leftarrow a$ ' is

$$\mathbf{Q}_{c \leftarrow a} = \begin{pmatrix} 1 & q_{c \leftarrow b}^{(56)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{E}'_{b \leftarrow a}}{\mathcal{E}_b} & \frac{\mathcal{E}_a}{\mathcal{E}_b} \end{pmatrix}$$

The matrices from the entrance of the complete multi stage system to a reference plane between stages follows from the recursive application of:

$$\mathbf{Q}_{c\leftarrow 0} = \mathbf{Q}_{c\leftarrow a} \mathbf{Q}_{a\leftarrow 0}$$
.

The total compression ratio from the entrance of the complete multi stage system to reference plane 'c' is $C_c = 1/q_{c \leftarrow 0}^{(55)}$. Consequently the compressed line charge density, current and wavenumber are $\Lambda_c = C_c \Lambda_0$, $I_c = C_c c \Lambda_0$ and $k_c = C_c k$. The line charge density and relative energy offset after the stage coincides formally with that at its entrance:

$$\Lambda_c(z_c) = \Lambda_c \Re \left\{ 1 + mG_c \exp(-jk_c z_c) \right\}$$

$$\eta_c(z_c) = \Re \left\{ \frac{\mathcal{E}'_c}{\mathcal{E}_c} z_c - jmH_c \exp(-jk_c z_c) \right\}$$

The gain G_c and energy modulation coefficient H_c are determined by the recursive relation

$$\begin{pmatrix} (G_c - 1)/k_c \\ H_c \end{pmatrix} = \mathbf{Q}_{c \leftarrow a} \begin{pmatrix} (G_a - 1)/k_a \\ H_a \end{pmatrix} - \frac{jI_a G_a Z_{b \leftarrow a}(ck_a)}{\mathcal{E}_b/e} \begin{pmatrix} q_{c \leftarrow b}^{(56)} \\ 1 \end{pmatrix} .$$
(32)

Fig. 29 shows the gain curves for the two stage system of the European XFEL that considers the linacs and impedances after ACC1 and the bunch compressors BC1 and BC2. Without uncorrelated energy spread the gain after BC2 exceeds about 10^5 for initial modulation wavelength $\lambda = 0.1$ mm.

d) Multi stages with uncorrelated energy spread

We consider the setup in Fig. 28 for an initial longitudinal phase space density $\psi(z_0, \eta_0) = \Lambda_0(z_0)F_\eta(\eta_0)$. A phase space point z_0 , η_0 at the entrance of the complete compression system is mapped to the entrance of a certain stage ' $c \leftarrow a$ ' as

$$\begin{pmatrix} z_a \\ \eta_a \end{pmatrix} = \mathbf{Q}_{a \leftarrow 0} \begin{pmatrix} z_0 \\ \eta_0 \end{pmatrix} + \Re \left\{ m \begin{pmatrix} g_a(\eta_0) \\ h_a(\eta_0) \end{pmatrix} \exp(-ikz_0) \right\}$$

and to the exit of the same stage as

$$\begin{pmatrix} z_c \\ \eta_c \end{pmatrix} = \mathbf{Q}_{c \leftarrow 0} \begin{pmatrix} z_0 \\ \eta_0 \end{pmatrix} + \Re \left\{ m \begin{pmatrix} g_c(\eta_0) \\ h_c(\eta_0) \end{pmatrix} \exp(-ikz_0) \right\} .$$

 $g_{a/c}(\eta_0)$ and $h_{a/c}(\eta_0)$ are auxiliary functions that vanish in the 0th reference plane and can be calculated for all other planes with the recursive equation:

$$\begin{pmatrix} g_c(\eta_0) \\ h_c(\eta_0) \end{pmatrix} = \mathbf{Q}_{c \leftarrow a} \begin{pmatrix} g_a(\eta_0) \\ h_a(\eta_0) \end{pmatrix} - \frac{I_a G_a Z_{b \leftarrow a}(k_a c)}{\mathcal{E}_b/e} \exp\left(-ik_a q_{c \leftarrow 0}^{(56)} \eta_0\right) \begin{pmatrix} q_{c \leftarrow b} \\ 1 \end{pmatrix}.$$
 (33)

The line charge density in plane c is calculated from the initial phase space density and the mapping equation by

$$\Lambda_c(\hat{z}_c) = \int \psi_0(z_0, \eta_0) \delta\left(z_c(z_0, \eta_0) - \hat{z}_c\right) dz_0 d\eta_0 \,,$$

or

$$\Lambda_c(\hat{z}_c) = \int \frac{\psi_0(z_0(\hat{z}_c, \eta_0), \eta_0)}{q_{c \leftarrow 0}^{(55)} + \Re \left\{ -ikmg_c(\eta_0) \exp(-iz_0(\hat{z}_c, \eta_0)) \right\}} d\eta_0 ,$$

where $z_0(z_c, \eta_0)$ is defined by the implicit equation

$$z_0 = \frac{z_c}{q_{c\leftarrow0}^{(55)}} - \frac{q_{c\leftarrow0}^{(56)}}{q_{c\leftarrow0}^{(55)}} \eta_0 - \Re \left\{ m \frac{g_c(\eta_0)}{q_{c\leftarrow0}^{(55)}} \exp(-ikz_0) \right\} .$$

This can be solved:

$$\Lambda_c(\hat{z}_c) = \Lambda_c \Re \left\{ 1 + mG_c \exp(-ik_c z_c) \right\} + \mathcal{O}(m^2)$$

The complex gain factor G_c follows from the integration to first order as:

$$G_c = \int (1 + ik_c g_c(\eta)) F_\eta(\eta) \exp\left(ik_c q_{c \leftarrow 0}^{(56)} \eta\right) d\eta \quad . \tag{34}$$

The successive application of Eqs. (33) and (34) determines the gain to all stages.

To reduce the large gain predicted in Fig. 29 it is foreseen to increase the uncorrelated energy spread. Supposed the spread $F_{\eta}(\eta)$ is gaussian with an rms value of 10 keV the maximal total gain is reduced to values below about 30 as shown in Fig. 30. Gain curves for the laser heater spectrum in Fig. 3 with the same rms value are plotted in Fig. 31. The maximal gain after compressor BC2 is kept below about 100.

3.2.3 Integral Equation Method

a) Method

A linear integral equation theory has been developed by [29] and [30] to calculate the amplification of microbunching. It considers transverse motion and dispersive trajectories but self effects are treated as offset independent. (Offset independent wakes and impedances see section 3.2.1 and especially section 3.2.1.c.) The initial four dimensional phase space distribution is

$$\psi_0(x_0, x'_0, z_0, \eta_0) = \psi_{\perp}(x_0, x'_0) \Lambda_0(z_0) \psi_{\eta}(\eta_0) \quad , \tag{35}$$

with the initial coordinates x_0, x'_0, z_0, η_0 and

$$\Lambda_0(z) = \Lambda_0 \Re \left\{ 1 + m \exp(-jkz) \right\} .$$

The longitudinal electrical field observed at longitudinal particle coordinate z at beam line position S is

$$E_{||}(z,S) = \Re \left\{ E_{||}(S) \exp(-jk(S)z) \right\}$$

with $E_{\parallel}(S)$ the complex amplitude and k(S) the local wave number as defined below. To first order in the modulation index *m* the initial point in phase space is mapped to

$$\begin{pmatrix} x_B \\ x'_B \\ z_B \\ \eta_B \end{pmatrix} = \mathbf{Q}_{B \leftarrow 0} \begin{pmatrix} x_0 \\ x'_0 \\ z_0 \\ \eta_0 \end{pmatrix} + \int_0^B dS \times \mathbf{Q}_{B \leftarrow S} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Re \left\{ \frac{E_{||}(S)}{\mathcal{E}(S)/e} \exp(-jk(S)z_S) \right\}$$
(36)

at the beam line position B. Note that the uppercase letters A, B and S are now real beam line positions. $\mathbf{Q}_{B\leftarrow A}$ is the linear (unperturbed) transformation from plane Ato B. The longitudinal density function is

$$\Lambda(z,S) = C(S)\Lambda_0 \Re \left\{ 1 + mG(S) \exp(-jk(S)z) \right\} + O(m^2)$$

with $C(S) = 1/q_{S \leftarrow 0}^{(55)}$ the local compression factor, k(S) = kC(S) the local wave number and $G(S) = G\{k, S_0 \rightarrow S\}$ the local gain. The modulated part of the density is related to the longitudinal electrical field by the impedance:

$$E_{||}(S) = -Z' \{ \omega = ck(S), S, ... \} \times c\Lambda_0 C(S) mG(S)$$

= $-Z'[S] \times c\Lambda_0 C(S) mG(S)$.

The local gain can be calculated as solution of the one dimensional integral equation

$$G(B) = G^{(0)}(B) + \int_{0}^{B} K(B,S) G(S) dS$$
(37)

with

$$G^{(0)}(B) = \int dx_0 dx'_0 d\eta_0 \times \psi_{\perp}(x_0, x'_0) \psi_{\eta}(\eta_0) \exp\left(jk \left(L(B)\right)^t \begin{pmatrix} x_0 \\ x'_0 \\ \eta_0 \end{pmatrix}\right)$$
(38)

the gain without longitudinal self fields (or without impedance), the kernel

$$K(B,S) = \frac{-jkq_{B\leftarrow S}^{(56)}}{q_{B\leftarrow 0}^{(55)}q_{S\leftarrow 0}^{(55)}} \frac{c\Lambda_0 Z'[S]}{\mathcal{E}(S)/e} \int dx_0 dx'_0 d\eta_0 \times \psi_{\perp}(x_0, x'_0)\psi_{\eta}(\eta_0) \exp\left(jk\left(L(B) - L(S)\right)^t \begin{pmatrix} x_0 \\ x'_0 \\ \eta_0 \end{pmatrix}\right)\right)$$
(39)

and

$$L(S) = \frac{1}{q_{S \leftarrow 0}^{(55)}} \begin{pmatrix} q_{S \leftarrow 0}^{(51)} \\ q_{S \leftarrow 0}^{(52)} \\ q_{S \leftarrow 0}^{(56)} \end{pmatrix} , \ Z'[S] = Z'\{\omega = ck(S), S, \ldots\} .$$

The transverse and energy parts of the phase space integrals can be separated and they can be integrated analytically for Gaussian density functions (see [29, 30] and appendix A).

b) Only Space Charge Effects - "SC model"

The integral equation method is a generalisation of the model of section 3.2.2 and can be used to derive an equivalent formulation. Therefore we distinguish ranges without and with transverse dispersion according to Fig. 28. Nondispersive parts are between S_{2m} and S_{2m+1} , dispersive sections range from S_{2m+1} to S_{2m+2} . The impedance in dispersive sections is set to zero, for the rest the same model is used as in discrete approach. The transverse and longitudinal phase spaces are not coupled by transport matrices if both boundary planes are in sections without transverse dispersion:

$$\mathbf{Q}_{B \leftarrow A} = \begin{pmatrix} q_{B \leftarrow A}^{(11)} & q_{B \leftarrow A}^{(12)} & 0 & 0\\ q_{B \leftarrow A}^{(21)} & q_{B \leftarrow A}^{(22)} & 0 & 0\\ 0 & 0 & q_{B \leftarrow A}^{(55)} & q_{B \leftarrow A}^{(56)}\\ 0 & 0 & q_{B \leftarrow A}^{(65)} & q_{B \leftarrow A}^{(66)} \end{pmatrix}$$

The transverse matrix elements are determined by the optics of the focusing lattice. The longitudinal matrix elements satisfy:

$$q_{B\leftarrow A}^{(55)} = q_{S_b\leftarrow S_a}^{(55)} ,$$
$$q_{B\leftarrow A}^{(56)} = q_{S_b\leftarrow S_a}^{(56)} \frac{\mathcal{E}(A)}{\mathcal{E}(S_a)} ,$$

$$q_{B\leftarrow A}^{(65)} = q_{S_b\leftarrow S_a}^{(65)} \frac{\mathcal{E}\left(S_b\right)}{\mathcal{E}\left(B\right)} ,$$

with $S_a < A < S_{a+1}$, $S_b < B < S_{b+1}$, a and b even. In sections without transverse dispersion the function $G^{(0)}(B)$ and the kernel of the integral equation simplify to

$$G^{(0)}(B) = \Psi_{\eta} \left(kq_{B \leftarrow 0}^{(56)} / q_{B \leftarrow 0}^{(55)} \right)$$
$$K(B,S) = \frac{-jkq_{B \leftarrow S}^{(56)}}{q_{B \leftarrow 0}^{(55)} q_{S \leftarrow 0}^{(55)}} \frac{c\Lambda_0 Z'[S]}{\mathcal{E}(S)/e} \Psi_{\eta} \left(k \left(\frac{q_{B \leftarrow 0}^{(56)}}{q_{B \leftarrow 0}^{(55)}} - \frac{q_{S \leftarrow 0}^{(56)}}{q_{S \leftarrow 0}^{(55)}} \right) \right)$$

with

$$\Psi_{\eta}(k) = \int d\eta \times \psi_{\eta}(\eta) \exp(jk\eta) \; .$$

The kernel K(B, S) vanishes if both coordinates are in the same section $S_b < S \leq B < S_{b+1}$ or if the source coordinate is inside of a dispersive section (according to our assumption that there is no impedance). Therefore the gain function G(B) is piecewise constant for $S_b < B < S_{b+1}$ as well as $G_b^{(0)}$. These constant gain factors $G_b^{(0)}$, G_b can be calculated from the matrix equation

$$(G_b) = \left(G_b^{(0)}\right) + (K_{b,a})(G_b) , \qquad (40)$$

with

$$\begin{split} G_{b}^{(0)} &= \Psi_{\eta} \left(kq_{S_{b} \leftarrow 0}^{(56)} / q_{S_{b} \leftarrow 0}^{(55)} \right) , \\ K_{b,a} &= \begin{cases} K_{b,a}^{+} & \text{for } b > a \\ 0 & \text{otherwise} \end{cases} , \\ K_{b,a}^{+} &= \frac{-jkq_{S_{b} \leftarrow S_{a}}^{(56)}}{q_{S_{b} \leftarrow 0}^{(55)} q_{S_{a} \leftarrow 0}^{(55)}} \frac{c\Lambda_{0}Z_{a}}{\mathcal{E}(S_{a})/e} \Psi_{\eta} \left(k \left(\frac{q_{S_{b} \leftarrow 0}^{(56)}}{q_{S_{b} \leftarrow 0}^{(55)}} - \frac{q_{S_{a} \leftarrow 0}^{(56)}}{q_{S_{a} \leftarrow 0}^{(55)}} \right) \right) , \\ Z_{a} &= \int_{S_{a}}^{S_{a+1}} Z'[S] dS . \end{split}$$

Formulations Eq. (34) and Eq. (40) are equivalent to each other.

c) Example: Dispersion in the Collimator

The bunch compression system (see Fig. 4) is followed by the main linac to 17.5 GeV and a collimation system that includes dispersive elements (see Fig. 5). The energy, transverse beam size and integrated impedance of the linac are plotted in Fig. 26.

Supposed the collimation system might be represented by a single dispersion coefficient $q_{\rm coll}^{(56)}$ the gain calculation has to consider three stages. This was assumed for the calculations in Figs. 32a and b. In Fig. 32a the influence of a small dispersion coefficient $q_{\rm coll}^{(56)}$ is investigated, that changes the total compression factor by only few percent.

Nevertheless the additional stage is of strong influence to the total microbunching gain and might increase it by a factor of about five for $q_{\text{coll}}^{(56)} = -0.6$ mm. This figure assumes on crest acceleration in the main linac so that the chirp is not altered. It is known that the cavity wake (of an unmodulated bunch) counteracts to the chirp and nearly compensates it. This effect is considered in Fig. 32b that assumes that the chirp is compensated by the rf system. The curves with and without chirp are nearly identical.

compensated by the rf system. The curves with and without chirp are nearly identical. Although the choice of a certain $q_{\rm coll}^{(56)}$ could lower the gain curves in the most important frequency range, it seems to be reasonable to avoid such compensations and to design a collimation system with low additional $q_{\rm coll}^{(56)}$. Fig. 33a shows the dispersion parameter from 0 (the beginning of the collimator) to a certain length L. The resulting gain curve of a system with more then 40 stages and space charge effects on all the drifts in the collimator is shown in Fig. 33b. The gain curve for this design is nearly unchanged by the collimation section. The space charge effects of the drifts on the energy level of 17.5 GeV are considered but negligible.

d) Only Coherent Synchrotron Radiation Effects - "CSR model"

This model is complementary to the "SC model" as it considers a CSR impedance in dispersive sections but no space charge effects. The CSR impedance and its steady state model

$$Z'\{\omega, S, \ldots\} = AZ_0 \sqrt[3]{\frac{\omega}{jc(R_{\rm curv}(S))^2}}$$

for bending magnets (with the curvature radius R_{curv}) has been discussed in section 3.2.1.c. Although it has been shown that the "switched" steady state impedance is a very rough approximation of the transient impedance, this model produces gain curves that are in good agreement to results obtained by direct particle tracking in presence of the renormalized longitudinal field according to Eq. (24) or simplified versions (f.i. [23, 31]). This has probably two reasons: fast oscillations of the impedance with the beam line coordinate S are not seen due to averaging effects and errors from the transients "into the magnet" cancel partially with errors from missing contributions after the magnet or "into the drift". In [32] an edge radiation impedance is used to consider contributions "after the magnet". A complete selfconsistent two dimensional model that does not artificially distinguish between "SC" and "CSR" effects is described and used in [33]. In this report we follow this approach and use the "switched" steady state impedance.

The application of Eqs. (37, 38, 39) uses a constant reference energy $\mathcal{E}(S) = \mathcal{E}$ and the transport matrix $\mathbf{Q}_{B\leftarrow A}$ regards the magnetic lattice of the compressor and a chirp generating matrix

$$\mathbf{Q}_{0+\leftarrow 0} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & c_h & 1 \end{pmatrix}$$

at its entrance as the initial distribution Eq. (35) is unchirped. The chirp coefficient c_h is chosen so that the compression factor $C = 1/(1 + c_h q_{\text{cmpr}}^{(56)})$ reaches the design value

 C_d with $q_{\text{cmpr}}^{(56)}$ the dispersion coefficient of the compressor. The required phase space integrals for gaussian initial density functions

$$\psi_{\perp}(x_0, x'_0) = \frac{1}{2\pi\varepsilon_0} \exp\left(-\frac{1}{2\varepsilon_0} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}^t \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}\right) ,$$
$$\psi_{\eta}(\eta_0) = \frac{1}{\sqrt{2\pi} \sigma_{\eta}} \exp\left(-\frac{1}{2} \left(\frac{\eta_0}{\sigma_{\eta}}\right)^2\right) ,$$

with ε_0 , α_0 , β_0 , γ_0 the initial twiss parameters and σ_η the uncorrelated energy spread, are listed in appendix A.

Microbunching gain curves for the first and independently for the second bunch compressor of the European XFEL have been calculated by the "CSR model". The gain curves for BC1 and different distribution functions and rms values of the uncorrelated energy spread are shown in Fig. 34a. The gaussian energy spread of 1 keV is typical for the source distribution without additional longitudinal heating. The maximal gain for that case is 4. If the energy spread is increased to 10 keV the gain is kept below 1.5 and drops for modulation wavelengths below 0.2 mm. The initial conditions for the second bunch compressor have been chosen according to the working point: the initial current is increased by the compression factor 20 of BC1 to 1 kA and due to conservation of phase space density the uncorrelated energy would be increased by the same factor. The gain curves are plotted in Fig 34b. These curve are quite similar to that of BC1 with compressed wavelengths.

e) SC and CSR Effects

The results of a one stage gain calculation with space charge and coherent radiation effects for the bunch compressors of the European XFEL can be seen in Figs. 35a and b. This calculation considers the space charge impedance in the preceding linac and the steady state CSR impedance in a certain compressor. For 10 keV uncorrelated energy spread the maximal gain is increased to about 10 in each of the compression stages. This demonstrates in comparison to Fig. 34 that micro bunch amplification is essentially driven by space charge effects.

3.2.4 Tracking Method

The one dimensional integral equation method avoids the discretization of phase space but the integration range increases with the total length of the system and the number of stages. Alternatively the perturbation in phase space can be tracked.

a) Method

The macro particle approach uses a set of N particles with initial charges q_n , initial positions

 $\mathbf{X}_{0,n} = \left(x_{0,n}, x_{0,n}', 0, \eta_{0,n}\right)^t$

and initial perturbations

 $\delta \mathbf{X}_{0,n} = \left(0, 0, \delta z_{0,n}, 0\right)^t$

for n = 0, 1, ..., N - 1. The unperturbed particles start in one slice and it is sufficient to track only these particles. Nevertheless we consider an arbitrary number R of slices that are equispaced in one period λ_0 . Therefore the 'slice' index r and the total index i = n + rN range from 0 to R - 1 and 0 to RN - 1 respectively. The properties of this extended particle set are

$$q_{n+rN} = q_n ,$$

$$\mathbf{X}_{0,n+rN} = \mathbf{X}_{0,n} + \lambda_0 \frac{r}{R} \mathbf{e}_z ,$$

$$\delta \mathbf{X}_{0,n+rN} = \Re \left\{ \delta \mathbf{X}_{0,n} \exp \left(j 2\pi \frac{r}{R} \right) \right\} .$$

The relations of the particle coordinates of the extended particle set to that of the 'slice' set for an arbitrary position S are

$$\mathbf{X}_{S,n+rN} = \mathbf{X}_{S,n} + \mathbf{e}_{z} \frac{2\pi}{k(S)} \frac{r}{R} ,$$

$$\delta \mathbf{X}_{S,n+rN} = \Re \left\{ \delta \tilde{\mathbf{X}}_{S,n} \exp \left(j \frac{2\pi}{k(S)} \frac{r}{R} \right) \right\} , \qquad (41)$$

with the complex perturbation $\delta \tilde{\mathbf{X}}_{S,n}$ that has to be calculated by tracking. Therefore we apply Eq. (36):

$$\begin{aligned} \mathbf{X}_{B,i} + \delta \mathbf{X}_{B,i} &= \mathbf{Q}_{B \leftarrow 0} \left(\mathbf{X}_{0,i} + \delta \mathbf{X}_{0,i} \right) \\ &+ \int_{0}^{B} dS \times \mathbf{Q}_{B \leftarrow S} \mathbf{e}_{\delta} \Re \left\{ \frac{E_{||}(S)}{\mathcal{E}(S)/e} \exp\left(-jk(S)(z_{S,i} + \delta z_{S,i})\right) \right\} \end{aligned}$$

to derive an update equation for the perturbation:

$$\delta \mathbf{X}_{B,i} = \mathbf{Q}_{B \leftarrow A} \delta \mathbf{X}_{A,i} + \int_{A}^{B} dS \times \mathbf{Q}_{B \leftarrow S} \mathbf{e}_{\delta} \Re \left\{ \frac{E_{||}(S)}{\mathcal{E}(S)/e} \exp\left(-jk(S)(z_{S,i} + \delta z_{S,i})\right) \right\} .$$

The term $\delta z_{S,i}$ in the exponential function contributes to second order and is therefore neglected:

$$\delta \mathbf{X}_{B,n+rN} = \mathbf{Q}_{B\leftarrow A} \delta \mathbf{X}_{A,n+rN} + \int_{A}^{B} dS \times \mathbf{Q}_{B\leftarrow S} \mathbf{e}_{\delta} \Re \left\{ \frac{E_{||}(S)}{\mathcal{E}(S)/e} \exp\left(-jk(S)(z_{S,n} + \frac{2\pi}{k(S)}\frac{r}{R})\right) \right\} .$$

Indeed this equation is consistent with Eq. (41) so that it is sufficient to apply the update equation to the complex perturbation of only one slice:

$$\delta \tilde{\mathbf{X}}_{B,n} = \mathbf{Q}_{B \leftarrow A} \delta \tilde{\mathbf{X}}_{A,n} + \int_{A}^{B} dS \times \mathbf{Q}_{B \leftarrow S} \mathbf{e}_{\delta} \Re \left\{ \frac{E_{||}(S)}{\mathcal{E}(S)/e} \exp\left(-jk(S)z_{S,n}\right) \right\} \quad . \tag{42}$$

For numerical integration in small steps $\Delta = B - A$ the integral $\int_A^B f(S) dS$ is approximated by $f(A)\Delta$. To obtain the modulation amplitude and the longitudinal field $E_{||}(S)$ the longitudinal charge density of "point particles" has to approximated by a continuous line charge density:

$$\Lambda(z,S) = \sum_{i} q_i \delta(z - (z_{,i} + \delta z_{,i})) = \Lambda(S) + \Re \left\{ \tilde{\Lambda}(S) \exp(-jk(S)z) \right\} + \dots$$

By Fourier analysis the modulation amplitude, electrical field and amplification follow to first order as:

$$\frac{\tilde{\Lambda}(S)}{\Lambda(S)} = \frac{jk(S)\sum_{n} q_n \delta z_{,n}}{\sum_{n} q_n} ,$$

$$E_{||}(S) = -Z'(\omega = ck(S), S) \times c\tilde{\Lambda}(S) ,$$

$$G(S) = \frac{1}{C(S)} \frac{\tilde{\Lambda}(S)}{\tilde{\Lambda}(0)} .$$
(43)

For numerical simulations the initial macro particle distribution was chosen on an equidistant grid in three phase space dimensions x, x', η and the macro charges q_n where set proportional to the local phase space density. The initial perturbation $\delta z_{0,n}^{(0)} = \delta_z$ assumes a constant displacement of all particles in the slice. As all nonlinear terms have been skipped the gain factor does not depend on the initial amplitude δ_z even if the calculated modulation amplitude gets larger than the unmodulated part.

b) SC and CSR Effects

The two stage bunch compression scenario of the European XFEL has been investigated by the tracking method. To obtain results comparable to the methods of sections 3.2.2 and 3.2.3 the same impedance models were used.

The curves in Fig. 36a show the gain after the first compression stage without impedances, with either space charge or CSR impedance and with both effects. The damping without impedances is caused by longitudinal smearing of the initial uncorrelated energy spread. This frequency dependency is proportional to the Fourier spectrum of the laser heater spectrum in Fig. 3. The curve with only the space charge impedance is in good agreement to the corresponding one stage gain in Fig. 31. The CSR and SC+CSR curves agree to Fig. 34a and 35b.

The curves in Fig. 36b show the gain from the exit of rf module ACC1 (130 MeV) to the exit of the second bunch compressor. The curve with only space charge effects agrees with that in Fig. 31. The gain for SC+SCR effects simultaneously is only slightly increased.

3.2.5 Amplification of Shot Noise

In the following the rms fluctuation of the compressed current due to initial shot noise is estimated. Therefore we represent the initial current i(t) by N discrete charges

$$i(t) = e \sum \delta(t - t_{\nu})$$

with ν from 0 to N-1. We neglect the sign of the particle charge and assume that the macroscopic distribution is sufficiently smooth and microbunching effects are only driven by the discrete structure of the distribution. The Fourier transformation is

$$I(\omega) = \int i(t) \exp(-j\omega t) dt = e \sum \exp(-j\omega t_{\nu}) .$$

The ensemble average $\langle |I(\omega)|^2 \rangle$ can be written as

$$\langle |I(\omega)|^2 \rangle = \int dt_0 ... dt_{N-1} \times e^2 \sum \exp(-j\omega(t_\nu - t_\mu))p(t_0) ... p(t_{N-1})$$

Both indices ν , μ of the double summation range from 0 to N-1. It is assumed that the probabilities p(t) for charges at arrival time t are independent and identical for all particles. Therefore the integration can be simplified to

$$\langle |I(\omega)|^2 \rangle = e^2 N + e^2 \sum_{\nu \neq \mu} \int dt_{\nu} dt_{\mu} \times \exp(-j\omega(t_{\nu} - t_{\mu}))p(t_{\nu})p(t_{\nu})p(t_{\mu})$$

and further to the well known result

$$\langle |I(\omega)|^2 \rangle = e^2 N + e^2 (N^2 - N) |P(j\omega)|^2$$

with $P(\omega)$ the Fourier transformation of p(t). The first term describes white noise, the second stands for the macroscopic shape of the current pulse. Note that the assumption of independent probabilities is violated if the longitudinal (or temporal) position of particles is altered by self effects causing microbunching. Therefore the ensemble average of current distributions in later reference planes of a bunch compression system might be even below e^2N .

We apply linear theory and assume that the shape part is linearly compressed while the white noise is amplified by the gain factor $\tilde{G}(\omega)$

$$\langle |I_C(\omega)|^2 \rangle \approx e^2 N |\tilde{G}(\omega)|^2 + e^2 (N^2 - N) |P(j\omega/C)|^2$$

with $\tilde{G}(\omega) = G\{\omega/(cC_B), S_0 \to S_B\}$ the gain factor for the initial wavenumber $\omega/(cC_B)$. The linear gain theory assumes a coasting beam or at least a probability function p(t) with very long and uniform flat top. We define the rms fluctuation $I_{\rm rms}$ that is superimposed to the flat top of the compressed pulse by the following equation

$$I_{\rm rms}^2 T = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^2 N |\tilde{G}(\omega)|^2 d\omega .$$

For this definition we assumed a long rectangular current pulse of duration T and applied Parseval's formula. With N = TI/e a direct relation between the compressed current $I = CI_0$, the gain function G and the rms fluctuation is found:

$$I_{\rm rms} = \sqrt{\frac{eI}{\pi} \int_{0}^{\infty} |\tilde{G}(\omega)|^2 d\omega} \quad . \tag{44}$$

The rms fluctuation for the gain curves in Fig. 36b with the total compression factor C = 100 and I = 5 kA are:

without self effects	0.1 A
only CSR effects	0.6 A
only SC effects	$28.3 \mathrm{A}$
SC and CSR effects	$29.9~\mathrm{A}$

These numbers assume white shot noise in the interface plane to the velocity independent high-energy model. Note that the curves in Fig. 36b are plotted as function of the modulation frequency before compression while the integral in Eq. (44) considers the frequency of the compressed modulation.

4 Low- and High-Energy Model

4.1 Linear Gain Model

Linear gain effects of the low energy part (gun to exit of first accelerating module) and the velocity independent part (after first accelerating module) are combined. Therefore we assume that an initial density modulation $\tilde{\Lambda}_0$ at the start of the low energy model is converted to a density and energy modulation with the amplitudes $\tilde{\Lambda}_A$ and $\tilde{\mathcal{E}}_A$ in the interface plane A between the models. These amplitudes are complex quantities and k_A is the wavenumber. The effects of these amplitudes are calculated separately and superimposed. The density modulation

$$\Lambda_A(z) = \Re \left\{ \Lambda_0 + \tilde{\Lambda}_A \exp(-ik_A z) \right\}$$

in the interface plane A is amplified to

$$\Lambda_{B,I}(z) = C_B \Lambda_0 \Re \left\{ 1 + \frac{\tilde{\Lambda}_A}{\Lambda_0} G_B \exp(-ik_B z) \right\}$$

in the interface plane B, supposed there is no initial energy modulation. Formally the effect of the energy modulation can be characterized by a function H_B that describes the conversion of $\tilde{\mathcal{E}}_A$ to the density modulation

$$\Lambda_{B,\mathcal{E}}(z) = C_B \Lambda_0 \Re \left\{ 1 + \tilde{\mathcal{E}}_A H_B \exp(-ik_B z) \right\}$$

in the interface plane B, supposed there is no initial density modulation. The sensitivity of density to initial energy modulation is proportional to the sensitivity of the gain function G_B to the very first impedance

$$Z_1 = \int\limits_{A}^{C} Z'(z) dz$$

that also produces energy modulation in the plane C at the entrance of the first dispersive element. Therefore the conversion function is

$$H_B = -\frac{1}{e} \frac{G_B - G_B^0}{Z_1 c \lambda_0} \tag{45}$$

with G_B^0 the gain function as calculated without the first impedance. The superposition of the effects of initial density and energy modulation lead to the distribution

$$\Lambda_B(z) = C_B \Lambda_0 \Re \left\{ 1 + \left(\frac{\tilde{\Lambda}_A}{\Lambda_0} G_B + \tilde{\mathcal{E}}_A H_B \right) \exp(-ik_B z) \right\}$$

and the total linear gain is

$$G_{\rm tot} = \frac{\tilde{\Lambda}_A}{\tilde{\Lambda}_0} G_B + \Lambda_0 \frac{\tilde{\mathcal{E}}_A}{\tilde{\Lambda}_0} H_B \quad . \tag{46}$$

4.2 Application to European XFEL

The gain and conversion functions G and H have been calculated from the exit of the first accelerating module to the exit of the optimized collimation system (compare Fig. 33a) with space charge fields in non dispersive sections. The results are plotted in Figs. 33b and 37. For instance for an initial wavelength of 0.1 mm the gain and conversion factor are $G \approx 20$ and $H \approx 3 \cdot 10^{-5}$ /eV. No initial density modulation but 1 keV energy modulation would cause 3% amplitude modulation at the exit.

The total gain from a reference plane 7 cm after the cathode to the end of the collimation section is calculated from the curves in Figs. 20, 33b, 37 and Eq. (46). The result is plotted in Fig. 38. G_I and G_E are the two terms of Eq. 46 and G_{tot} is the total gain. For wavelengths between 0.7 and 1.2 mm the total gain is between 8 and 12 and is essentially driven by energy modulations. To achieve a fluctuation of the compressed current of few percent, the fluctuation of the current in the gun has to be of the order of permille for that range of wavelengths.

It has to be noted that the total gain is lower than that of the high energy part. Therefore the rms current fluctuation calculated from Eq. (44) for white shot noise in the interface plane to the high energy part might be over estimated. Fig. 38 covers only a part of the frequency range contributing to the integral.

5 Conclusion and Summary

The amplification of micro modulations has been investigated for particle beams with low transverse emittance and high longitudinal density as they are required for FEL systems. A low energy regime starting from the gun is distinguished from a high energy regime where velocity dependent effects are negligible. The low energy regime was investigated by numerical methods. For the high energy regime linear theories are summarized and extended. The methods are applied to the European XFEL.

The numerical simulation of the low energy part by ASTRA starts at the photo cathode and considers all effects up to the end of the first accelerating module at the energy level of ~ 130 MeV. Therefore the complex interaction of external fields and self fields are taken into account as well as non linearities. The stimulation of micro modulations and the analysis was chosen to investigate linear effects. The micro modulation was created few centimetres after the cathode by a small manipulation of the longitudinal particle position. Extensive convergence studies have been done for modulation wavelengths between 0.2 and 1.2 mm. (The lower limit is due to the numerical effort for simulations with high resolution, the upper is due to the finite bunch length and field non linearities.) Plasma oscillations have been observed and and an artificial model was used to study numerical properties of the simulation. For the considered wavelength range the initial modulation of 5% was suppressed to less that 1% with a zero crossing in the middle of the interval. The amplitude of energy modulation is about 5 keV for long wavelength decreasing to about 1 keV for the lowest wavelength.

Micro bunching in the high energy part is due to longitudinal self fields before and in dispersive sections. Important sources of self fields for short wavelengths are the SC and CSR impedance. The SC impedance is considered for linear sections without dispersion. (The longitudinal bunch population is frozen.) A steady state CSR impedance (that excludes SC contributions) is used for bunch compressor chicanes. Effects in a multi stage system of sections with longitudinal dispersion (usually bunch compressors) in linear sections (accelerators and diagnostic) are described by linear theory. The micro bunching gain calculation can be simplified and reduced to a discrete system if impedances in dispersive sections are negligible. An efficient scheme for such calculations has been described. Otherwise a one dimensional integral equation has to be solved or particles have to be tracked.

The theory was applied to single and multi stage setups with and without uncorrelated energy spread close to the European XFEL. Without uncorrelated energy spread the amplification of micro modulations can be many orders of magnitude (10^5 for $\lambda = 0.1$ mm) by space charge effects alone. This can be reduced if the uncorrelated energy spread is increased by a laser heater. Results are shown for a gaussian spectrum and for a realistic laser heater spectrum, both with the rms energy spread of about 10 keV. The one stage gain in bunch compressors due to CSR (without SC) is below 1.5. The total gain in the high energy part is kept below about 100 for SC and CSR impedances. For large values the gain curve with only SC effects agrees good with that for both impedances. Therefore fast and efficient optimizations with the discrete scheme (based on only SC impedances) are possible. The results of calculations for the low and high energy part have been combined. Although the density modulation from the gun is strongly reduced in the interface plane, the co-existing energy modulation causes a gain below 12 in the given frequency range. The maximum is for $\lambda = 0.9$ mm.

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A Some phase space integrals

The analytic solutions of the phase space integrals in Eqs. (38) and (39) are calculated for gaussian transverse and longitudinal density distributions:

$$\psi_{\perp}(x_0, x'_0) = \frac{1}{2\pi\varepsilon_0} \exp\left(-\frac{1}{2\varepsilon_0} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}^t \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}\right) ,$$
$$\psi_{\eta}(\eta_0) = \frac{1}{\sqrt{2\pi} \sigma_{\eta}} \exp\left(-\frac{1}{2} \left(\frac{\eta_0}{\sigma_{\eta}}\right)^2\right) .$$

The three dimensional integral in Eq. (38) is separated to the product of $I_1(B)$ and $I_2(B)$ with

$$I_1(B) = \int dx_0 dx'_0 \times \psi_{\perp}(x_0, x'_0) \exp(jk \left(L_1(B)x_0 + L_2(B)x'_0\right)) ,$$

$$I_2(B) = \int d\eta_0 \times \psi_{\eta}(\eta_0) \exp(jk L_3(B)\eta_0) ,$$

with the solutions:

$$I_{1}(B) = \exp\left(-\frac{\varepsilon_{0}k(B)^{2}}{2} \begin{pmatrix} q_{B\leftarrow0}^{(51)} \\ q_{B\leftarrow0}^{(52)} \\ q_{B\leftarrow0}^{(52)} \end{pmatrix}^{t} \begin{pmatrix} \beta_{0} & -\alpha_{0} \\ -\alpha_{0} & \gamma_{0} \end{pmatrix} \begin{pmatrix} q_{B\leftarrow0}^{(51)} \\ q_{B\leftarrow0}^{(52)} \\ q_{B\leftarrow0}^{(52)} \end{pmatrix} \right) ,$$

$$I_{2}(B) = \exp\left(-\frac{1}{2} \left(q_{B\leftarrow0}^{(56)}k(B)\sigma_{\eta}\right)^{2}\right) .$$

The three dimensional integral in Eq. (39) is of the same type and separated to the product of $I_3(B, S)$ and $I_4(B, S)$ with

$$I_3(B,S) = \int dx_0 dx'_0 \times \psi_{\perp}(x_0, x'_0) \exp(j (\Delta_1 x_0 + \Delta_2 x'_0)) ,$$

$$I_4(B,S) = \int d\eta_0 \times \psi_{\eta}(\eta_0) \exp(j \Delta_3 \eta_0) ,$$

$$\Delta = (L(B) - L(S))k ,$$

and the solutions:

$$I_{3}(B,S) = \exp\left(-\frac{\varepsilon_{0}}{2} \left(\begin{array}{c} \Delta_{1} \\ \Delta_{2} \end{array}\right)^{t} \left(\begin{array}{c} \beta_{0} & -\alpha_{0} \\ -\alpha_{0} & \gamma_{0} \end{array}\right) \left(\begin{array}{c} \Delta_{1} \\ \Delta_{2} \end{array}\right)\right) ,$$
$$I_{4}(B,S) = \exp\left(-\frac{1}{2} \left(\Delta_{3}\sigma_{\eta}\right)^{2}\right) .$$



Figure 1: 3D sketch of the injector with one rf gun and TESLA module (yellow) per beamline. The rf gun and bending magnets are drawn in blue.



Figure 2: LCLS laser heater. From [5].



Figure 3: Normalized uncorrelated energy spectra, solid: laser heater with typical parameters for 10 keV rms, dashed: gaussian distribution.



Figure 4: 3D sketch of the two stage bunch compression system. The modules of the third harmonic rf (red) are upstream of the first bunch compressor.



Figure 5: 3D sketch and block diagram of the accelerator complex. The length of the main linac of about 1.2 km is suppressed. The length of the collimation system is about 230 m.



Figure 6: Wavelength of plasma oscillations for I = 50 A and typical impedances.



Figure 7: Sketch of rf gun.



Figure 8: Solenoid field.



Figure 9: Cavity fields along the first 14.5 m of the European XFEL.



Figure 10: Relative current modulation in % for the plasma oscillation, 2D space charge model, for different longitudinal resolutions $N_{long} = 100, 200, 400, 800$ steps.



Figure 11: Energy modulation for the plasma oscillation, 2D space charge model, for different longitudinal resolutions $N_{long} = 100, 200, 400, 800$ steps.



Figure 12: Relative current modulation in % for the plasma oscillation, 3D space charge model, for different longitudinal resolutions $N_z = 64, 128, 256, 512$ steps.



Figure 13: Energy modulation for the plasma oscillation, 3D space charge model, for different longitudinal resolutions $N_z = 64, 128, 256, 512$ steps.



Figure 14: Relative current modulation for $\lambda = 1.0$ mm, simulations with 500,000 macro particles ($N_z = 64$) and 5 million macro particles ($N_z = 256, 512$).



Figure 15: Energy modulation for $\lambda = 1.0$ mm, simulations with 500,000 macro particles ($N_z = 64$) and 5 million macro particles ($N_z = 256, 512$).



Figure 16: Relative current modulation for $\lambda = 0.75$ mm, simulations with 500,000 macro particles ($N_z = 64$) and 5 million macro particles ($N_z = 256, 512$).



Figure 17: Energy modulation for $\lambda = 0.75$ mm, simulations with 500,000 macro particles ($N_z = 64$) and 5 million macro particles ($N_z = 256, 512$).



Figure 18: Relative current modulation along the first 14.5 m of the European XFEL.



Figure 19: Energy modulation along the first 14.5 m of the European XFEL.



Figure 20: Amplitude and energy modulation after the first accelerating module due to a amplitude modulation of 5% at 7 cm after cathode.



Figure 21: An illustration of microbunching instability in a bunch compressor. From [5].



Figure 22: Impedance *T*-functions and their low frequency approximations as used in Eq. (20) for a beam with round transverse cross section. $T_h(x)$ hollow beam, $T_p(x)$ pencil beam, $T_g(x)$ gaussian beam.



Figure 23: Transient CSR impedance vs frequency for the transition (a) from a straight section into an arc, (b) from an arc into a straight section. Radius of arc $R_0=10$ m, relativistic factor $\gamma \to \infty$. Parameter is the length after transition. For comparison the steady state impedance Eq. (25) in the arc is plotted.



Figure 24: Transient CSR impedance vs beam line coordinate for the transition from a straight section into an arc with radius $R_0=10$ m and length 0.5 m followed by a drift for $\gamma \to \infty$. Real and imaginary part are compared with the steady state impedance Eq. (25) for (a) f = 0.3 THz, (b) f = 1 THz and (c) f = 3 THz.



Figure 25: Impedance per length Z' for different positions in the European XFEL. The frequency window indicates the range of of initial modulation with wavelength λ between 0.1 and 1 mm. (a) 130 MeV, (b) 500 MeV, due to longitudinal compression the window is shifted to higher frequencies, (c) 2 GeV, 2nd longitudinal compression and (d) 17.5 GeV, after main linac. (black solid) cavity impedance, (red) SC impedance, free space, (blue) SC impedance with perfect conducting beam pipe and (green) SC impedance with copper beam pipe.

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(a) from exit of injector (130 MeV) to entrance of BC1 (500 MeV)



(c) from exit BC2 to the entrance of the collimator (17.5 GeV)

Figure 26: Relativistic factor γ , typical rms beam radius σ_r vs beam line coordinate and impedance Z vs frequency, integrated for different sections of the European XFEL.



Figure 27: Gain curves: single stage end of ACC1 to end of BC1.



Figure 28: Multi stage bunch compression system.



Figure 29: Gain curves: two stage without uncorrelated energy spread.



Figure 30: Gain curves: two stage with gaussian uncorrelated energy spread.



Figure 31: Gain curves: two stage with laser heater uncorrelated energy spread.



Figure 32: Gain curves of a three stage system to the end of the collimation section, with uncorrelated energy spread of 10 keV rms by a laser heater.



Figure 33: Dispersion and gain curves in an optimized collimation section.



Figure 34: One stage gain curves. (a) BC1 at 500 MeV compression from 50 A to 1 kA with $q_{56}=103$ mm, $C_1 = 20$; (b) BC2 at 2 GeV compression from 1 kA to 5 kA with $q_{56}=17.4$ mm.



Figure 35: One stage gain due to SC impedance in the linac and CSR impedance in the bunch compressor. (a) Linac from 130 MeV to 500 MeV and BC1; (b) Linac from 500 MeV to 2 GeV and BC2.



Figure 36: Gain curves: SC and/or CSR gain from 130MeV to end of BC1 or BC2. With laser heater, 10 keV.



Figure 37: Current modulation du to initial energy modulation.



Figure 38: Gain from amplitude modulation at 7 cm after cathode to BC2 exit.