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SLATER'S THEOREM FOR MAGNETIC MIRROR BOUNDARY PERTURBATIONS

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Abstract
A general form of Slater's formula, taking into account perturbations of Electric as well as Magnetic Mirror boundaries is presented. The frequency shift induced by the Magnetic Mirror perturbation has the opposite sign to that of the Electric one.

Introduction

The procedure shown in Fig. (1) is often used to compute frequency shifts induced by "small" deformations of a cavity [1] under the effects of various forces: helium bath pressure, thermal stresses, tuning [2] or Lorentz forces [3].

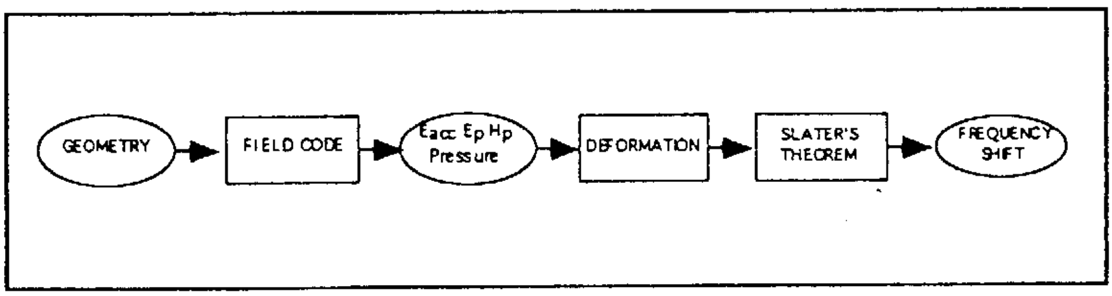

FIG. 1 - Procedure to compute "small" frequency shifts.

Such scheme is very useful to evaluate the effect of "small" perturbations of the cavity shape because it provides good and fast estimates of the frequency shift without need for time consuming field codes that may moreover introduce errors of the same order of the frequency shift itself because of their finite discretization of boundary walls.

The computational scheme is however not as simple as it appears at a first glance. For instance a not completely rigorous application of Slater's theorem may produce an overestimate of the frequency shift, in particular when a cavity oscillating in a π - mode is shortened under the effects of the perturbation. Recall in particular that calculating the frequency of an accelerating π – mode with a field code requires imposing Magnetic Mirror boundary conditions on the iris planes.

We discuss in the next section a general derivation of Slater's theorem taking into account both Electric (EM) and Magnetic Mirror (MM) boundary conditions.

Discussion of Slater's theorem

The frequency shift of a resonant mode under the effect of a perturbation of the cavity wall is computed from the well known Slater's formula [4]:

$$\frac{\delta\omega_v}{\omega_v} = \frac{1}{4\overline{U}} \int_{\delta v^*} \left(\varepsilon_o E^2 - \mu_o H^2\right) dv \tag{1}$$

where:

$$\overline{U} = \frac{1}{4} \int_V \left(\varepsilon_o E^2 + \mu_o H^2\right) dv \tag{2}$$

is the average energy stored in the cavity volume V and δv^* is the volume variation caused by the deformation at the neighborhood surface S representing the cavity wall. It is important to stress that S is a perfectly conducting wall, on which EM boundary condition: $\mathbf{n} \times \mathbf{E} = 0$ applies.

The formula is incorrect for all perturbations of a MM boundary S' where $\mathbf{n} \times \mathbf{H} = 0$, as for example in the case of a deformation of the iris plane considering the $TM^{010}{}_{\pi}$ accelerating mode. This situation can occur for perturbations leading to shortening of the cavity, see Fig. (2).

A more general Slater's formula, taking into account the pertubation of EM as well as MM boundary and also the possible existence of a beam current density **J** is the following [5]:

$$\frac{\delta\omega_v}{\omega_v} = \frac{1}{4\overline{U}} \int_{\delta v^*} \left(\varepsilon_o E^2 - \mu_o H^2\right) dv + \frac{1}{4\overline{U}} \int_{\delta v^{**}} \left(\mu_o H^2 - \varepsilon_o E^2\right) dv + \frac{j}{4\overline{U}\omega_v} \int_V \left(\mathbf{J} \cdot \mathbf{E}^*\right) dv \tag{3}$$

The first term coincide with eq. (1), the second is valid on the iris plane and has the opposite sign than the first , the third accounts for the beam loading effects.

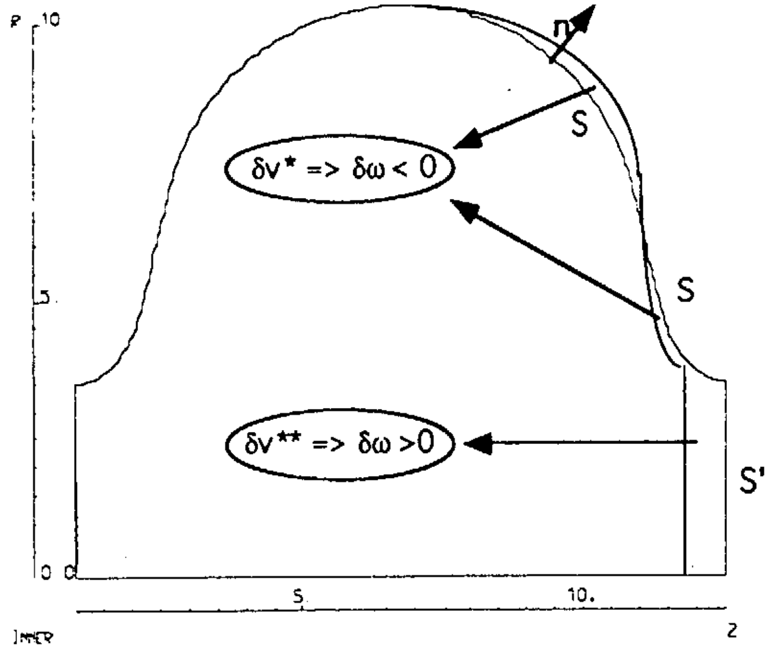

FIG. 2 - A schematic drawn of a cavity perturbation. The Slater's theorem must be applied not only in regions δv^* where $\delta\omega < 0$ but also in region δv^{**} where $\delta\omega > 0$ in the accelerating mode.

To demonstrate eq. (3), we perform a modal expansion of the wave equation.

Let **E** and **H** be written in terms of the cavity field functions $\{\mathbf{e}_v,\mathbf{h}_v\}$ within a volume V as follows:

$$\mathbf{E}(x,y,z,t)=\Sigma_v a_v(t)\mathbf{e}_v(x,y,z)e^{j\omega_v t} \qquad \mathbf{H}(x,y,z,t)=\Sigma_v b_v(t)\mathbf{h}_v(x,y,z)e^{j\omega_v t} \tag{4}$$

where $\{\mathbf{e}_v,\mathbf{h}_v\}$ must satisfy the following equations:

$$\left[\nabla^2 + k_v^2\right]\mathbf{e}_v=0 \qquad \left[\nabla^2 + k_v^2\right]\mathbf{h}_v=0 \tag{5}$$

within a volume V, where $k_v=\omega_v/c$. For solenoidal fields we have:

$$\nabla\bullet\mathbf{e}_v=0 \qquad \nabla\bullet\mathbf{h}_v=0 \tag{6}$$

and the boundary conditions are (EM over a surface S and MM over another surface S'):

$$\mathbf{n}\times\mathbf{e}_v=0 \;\text{ over S} \qquad \mathbf{n}\times\mathbf{h}_v=0 \;\text{ over S'} \tag{7a}$$

$$\mathbf{n}\cdot\mathbf{h}_v=0 \;\text{ over S} \qquad \mathbf{n}\cdot\mathbf{e}_v=0 \;\text{ over S'} \tag{7b}$$

where **n** is a unit vector pointing outwards from the cavity surface. The normalization conditions are:

$$\int_V \mathbf{e}_v\cdot\mathbf{e}_\mu^*\,dv = \delta_{v\mu} \qquad \int_V \mathbf{h}_v\cdot\mathbf{h}_\mu^*\,dv = \delta_{v\mu} \tag{8}$$

and the coefficients $a_v(t)$, $b_v(t)$ in eq. (4) are given by:

$$a_v = \int_V \mathbf{E}\cdot\mathbf{e}_v{}^*\,dv \qquad\qquad b_v = \int_V \mathbf{H}\cdot\mathbf{h}_v{}^*\,dv \tag{9}$$

From Maxwell's equations we have:

$$\nabla \times \mathbf{e}_v = k_v\mathbf{h}_v \qquad\qquad \nabla \times \mathbf{h}_v = k_v\mathbf{e}_v \tag{10}$$

and the average stored energy is:

$$\overline{U} = \frac{1}{2}\epsilon_o\Sigma_v|a_v|^2 = \frac{1}{2}\mu_o\Sigma_v|b_v|^2 \tag{11}$$

From the wave equation

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_o\frac{\partial\mathbf{J}}{\partial t} \tag{12}$$

we obtain for each component of the field amplitude the following relationship:

$$\begin{aligned}\frac{1}{c^2}\frac{d^2a_v}{dt^2} + k_v^2a_v = -\mu_o\frac{d}{dt}&\left[\int_V\left(\mathbf{J}\cdot\mathbf{e}_v^*\right)dv - \int_{\delta s^*}(\mathbf{n}\times\mathbf{H})\cdot\mathbf{e}_v^*ds\right] + \\ &-k_v\int_{\delta s^{**}}(\mathbf{n}\times\mathbf{E})\cdot\mathbf{h}_v^*ds\end{aligned} \tag{13}$$

where δs^* and δs^{**} are closed surfaces obtained joining the perturbed and the un-perturbed surfaces of the EM and MM type respectively. In the last equation we take into account the contributions of the surface current $\mathbf{J}_e=\mathbf{n}\times\mathbf{H}$ (second integral) and of the fictious magnetic surface current $\mathbf{J}_m=\mathbf{n}\times\mathbf{E}$ (third integral), which appear at the discontinuity in the tangential component of the fields $\mathbf{H}$ and $\mathbf{E}$ respectively, on the perturbed surfaces (see [4] chapter IV). The first integral accounts for the contribution of the beam current.

Setting now

$$\mathbf{E} = a_v(t)\mathbf{e}_v \qquad\qquad \mathbf{H} = b_v(t)\mathbf{h}_v \tag{14}$$

the surface integrals in (13) become:

$$\begin{aligned}\int_{\delta s^{**}} (\mathbf{n}\times a_v\mathbf{e}_v)\cdot\mathbf{h}_v^* ds &= a_v \int_{\delta s^{**}} \mathbf{n}\cdot(\mathbf{e}_v\times\mathbf{h}_v^*)ds \\ &= a_v \int_{\delta v^{**}} \nabla\cdot(\mathbf{e}_v\times\mathbf{h}_v^*)dv \\ &= a_v \int_{\delta v^{**}} (\mathbf{h}_v^*\cdot\nabla\times\mathbf{e}_v - \mathbf{e}_v\cdot\nabla\times\mathbf{h}_v^*)dv \\ &= a_v \int_{\delta v^{**}} (\mathbf{h}_v^*\cdot k_v\mathbf{h}_v - \mathbf{e}_v\cdot k_v\mathbf{e}_v^*)dv \\ &= k_v a_v \int_{\delta v^{**}} (h_v^2 - e_v^2)dv \end{aligned} \tag{15}$$

where we applied the divergence theorem to the volume δv^{**} enclosed by the surface δs^{**} and:

$$\begin{aligned}\int_{\delta s^{*}} (\mathbf{n}\times b_v\mathbf{h}_v)\cdot\mathbf{e}_v^* ds &= k_v b_v \int_{\delta v^{*}} (e_v^2 - h_v^2)dv \\ &= j\omega_v\varepsilon_0 a_v \int_{\delta v^{*}} (e_v^2 - h_v^2)dv \end{aligned} \tag{16}$$

where δv^* is the volume enclosed by the surface δs^* and we have used the equality $k_v b_v = j\omega_v\varepsilon_0 a_v$ derived from Maxwell's equations for a source free cavity.
Introducing the above integrals in (13), calculating the temporal derivatives $da_v/dt=j\omega a_v$ (where ω is the frequency of the perturbed cavity) and multiplying both sides of (13) by a_v^*, ($a_v \cdot a_v^* = |a_v|^2$), we get:

$$\begin{aligned}\left[\omega_v^2-\omega^2\right]|a_v|^2 = -\omega_v^2|a_v|^2 \int_{\delta v^{*}} (e_v^2 - h_v^2)dv - \omega_v^2|a_v|^2 \int_{\delta v^{**}} (h_v^2 - e_v^2)dv + \\ -j\omega\mu_0 c^2 \int_{V} (\mathbf{J}\cdot\mathbf{E}^*)dv \end{aligned} \tag{17}$$

To first order in $\delta\omega_v$, setting $\omega=\omega_v+\delta\omega_v$ we finally obtain:

$$\frac{\delta\omega_v}{\omega_v} = \frac{1}{2}\int_{\delta v^*}\left(e_v^2 - h_v^2\right)dv + \frac{1}{2}\int_{\delta v^{**}}\left(h_v^2 - e_v^2\right)dv + \\ + \frac{j}{2\varepsilon_d |a_v|^2 \omega_v}\int_V \left(\mathbf{J}\cdot\mathbf{E}^*\right)dv \tag{18}$$

that because of (11) and (14) is identical with (3).

Example

We have computed frequency and fields of a TESLA cavity with an improved version of SUPERFISH that performs a discretization of the Helmoltz equation over an irregular triangular mesh of up to 32000 points. The electric surface field amplitude is computed from the relation: $E=(1/kr)\partial(rH)/\partial\ell$, ℓ being the path coordinate along the cavity boundary. Because the quantity $rH(r,z)$ is discretized up to second order on the mesh points, a three point numerical derivation of the electric field was used to obtain the field to second order also [1].

As an example of the importance of using the complete Slater formula (3), note that shortening the TESLA cavity by 10^{-3} mm, by cutting off a slice at the iris plane - where Magnetic Mirror condition have been imposed - while leaving the rest of the cavity unchanged, gives according to eq. (3) a frequency shift of +2.031 KHz in agreement with SUPERFISH.

This result is compared in Fig. (3) to the frequency shifts induced by a "cut" of the same size on a Electric Mirror boundary of the same TESLA cavity.

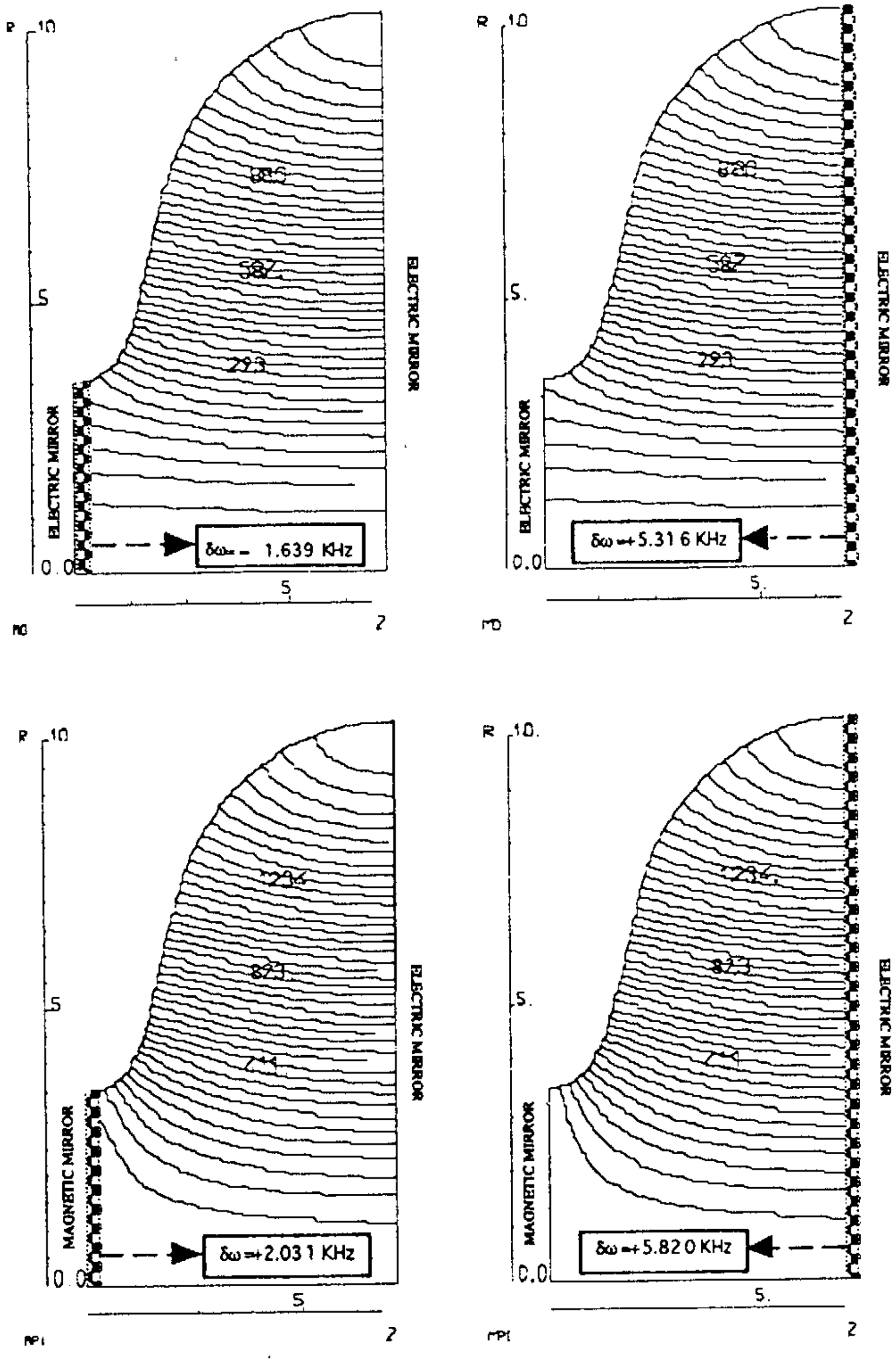

Fig. 3 - Generalized Slater's formula (3) results: frequency shifts induced by shortening the TESLA cavity by 10^{-3} mm, by cutting off a slice at either the iris or at the equator plane - where different boundary condition are imposed - leaving the rest of the cavity unchanged. (Unperturbed frequencies are: 0_TM_{010} = 1276.667 MHz and π_TM_{010} = 1301.016 MHz).

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