LINAC BASED FREE-ELECTRON LASER

J. Rossbach
Universität Hamburg, Hamburg, Germany

Abstract
A basic treatment of the principle of the linac-driven free-electron laser (FEL) is given. The first part of the paper describes the FEL in low-gain approximation, and in the second part the high-gain FEL theory is given. The majority of the treatment describes FELs in one dimensional approximation, neglecting effects by diffraction of radiation and by electron beam emittance. Only in the final section a few remarks on these issues are given. The ambition of the paper is by no means any progress in FEL theory but a clear presentation of basic FEL theory concepts with explicit derivation of the formulae from first principles.

INTRODUCTION

The basic theory of linac-driven free-electron lasers (FEL) presented in this paper is based on lectures given for the CERN Accelerator School Course on “Synchrotron Radiation and Free Electron Lasers” 2-9 July 2003 in Brunnen, Switzerland. The intention of the paper is neither a report on progress in FEL theory nor a complete and in-depth treatment of the subject. It is rather an attempt to present the basic concepts of linac-based FELs starting from first principles and deriving formulae explicitly step by step so that students should be able to follow without doing long derivations and calculations by themselves. For the sake of simplicity the FEL theory is given in one-dimensional approximation, i.e. only longitudinal electron motion is considered and diffraction effects of radiation are neglected. This approximation is particularly justified for FELs operating in the VUV- or X-ray wavelength regime because

- space charge effects are typically of less importance at ultra-relativistic energies typical for such kind of radiation sources.
- the fundamental, coaxial mode typically dominates the radiation in the high-gain regime of FELs, which is of particular interest for such short wavelengths.

The paper covers the material presented during the two one-hour lectures plus a few remarks (hopefully) useful for the student, but nothing beyond. The MKSA (or “practical”) system of units and a right-handed Cartesian coordinate system (with *z* being the longitudinal coordinate) are used throughout the paper. [1, 2]

1. THE FREE-ELECTRON LASER IN LOW GAIN APPROXIMATION

1.1 Radiation power of a point-like electron distribution moving at ultra-relativistic speed

An FEL is basically a classical device, i.e., with very few exceptions, all features can be derived and described by classical electrodynamics and relativistic kinematics. Thus, as an introduction to the principle of FELs, it is useful to recall some basics of classical electrodynamics [3].

Let us consider an electric charge q moving at ultra-relativistic speed with respect to the laboratory system. Classical electrodynamics says that any accelerated charge emits electromagnetic radiation. The radiation power P_γ emitted by a charge q accelerated at $\dot{v}^*$ is given by

$$P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\dot{v}^*\right)^2 = P_\gamma^* \tag{1}$$

ε_0 is the electric permittivity of vacuum and c is the speed of light. The asterisk * means that the respective quantity is to be evaluated in a system * moving along with the charge such that its velocity v^* is much smaller than c.

Eq. (1) makes the important statement that the power P_γ observed in *any* system is the same as the power P_γ^* calculated in the co-moving system in the way given by Eq. (1)! This fact makes is easy to calculate the radiation power observed in the laboratory system in terms of quantities measured in the lab system: We just have to express the acceleration $\dot{v}^*$ by quantities measured in the lab system. This is accomplished by the Lorentz transformation of acceleration given by (see Ref. [3] p. 47 ff)

$$\dot{v}_z^* = \gamma^3 \dot{v}_z, \quad \dot{v}_y^* = \gamma^2 \dot{v}_y \;, \; \dot{v}_x^* = \gamma^2 \dot{v}_x \tag{2}$$

with $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $\beta = \dfrac{v_0}{c}$. The velocity v_0 of the moving system * with respect to the lab system is assumed to be in the z-direction, see Fig. 1.

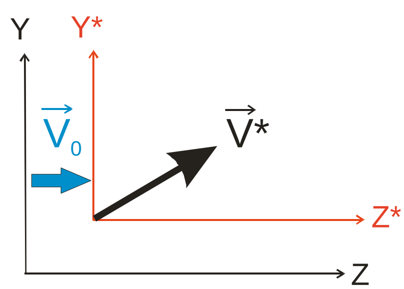

Fig. 1 Definition of a coordinate system denoted with an asterisk * moving with speed v_0 with respect to the laboratory system.

It is important to realize from Eq. (1) that the component $\dot{v}_z$ of acceleration parallel to the velocity of the moving system transforms in a different way than the components perpendicular to it. Acceleration perpendicular to the relativistic motion of the electron beam is the only one of practical relevance, because it is achieved by the motion of the electrons in presence of an external magnetic field. For the case of vertical acceleration, for example, Eq. (1) reads

$$P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 \dot{v}_y^2 \;, \tag{3}$$

indicating that one gains a large increase in radiation power when accelerating the electron beam to ultra-relativistic (i.e. $\gamma >> 1$) energies.

In terms of the FEL principle, the most important consequence of Eq. (1) is that the radiation power scales quadratically with the charge. Taking into account that, in practice, the charge consists of a large number N of electrons with elementary charge e_0, Eq. (3) can be written in the form

$$P_\gamma = \frac{N^2 e_0^2}{6\pi\varepsilon_0 c^3}\gamma^4 \dot{v}_y^2 \ . \tag{4}$$

Obviously, the radiation power per electron is $P_\gamma(\text{each electron}) = \frac{P_\gamma}{N} = \frac{N e_0^2}{6\pi\varepsilon_0 c^3}\gamma^4 \dot{v}_y^2$, i.e. it is N times larger that the radiation power of a single electron (i.e. not accompanied by many others) $P_\gamma(\text{single electron}) = \frac{e_0^2}{6\pi\varepsilon_0 c^3}\gamma^4 \dot{v}_y^2$. This is, because the electrons moving in a bunch have to perform work against the electric field generated by the co-moving electrons. This can be considered the classical analogon to stimulated emission.

The main condition for Eq. (4) to hold is that all N electrons have to make up a "point-like" charge distribution. For a radiating bunch of electrons moving at ultra-relativistic speed this means, in particular, that the longitudinal dimension of the bunch must be shorter than the radiated wavelengths. For wavelengths much shorter than the visible, this is difficult (or impossible) to achieve. In conventional synchrotron radiation sources like electron storage rings, for instance, the radiation wavelength attractive for users is in the Nanometer range (or below), while the sizes of electron bunches in storage rings is a few Millimeters typically. As a consequence, the radiation power of a bunch of N electrons in a storage ring is only $N \cdot P_\gamma(\text{single electron})$: All electrons radiate independent of each other (incoherent radiation). Obviously, there is a factor of N (which is huge indeed) to be regained, if only there was a mechanism to rearrange the electrons on the scale of the optical wavelength. The FEL principle provides such a mechanism.

Fig. 2 shows schematically the key components of a free-electron laser using an electron beam accelerated by a linear accelerator.

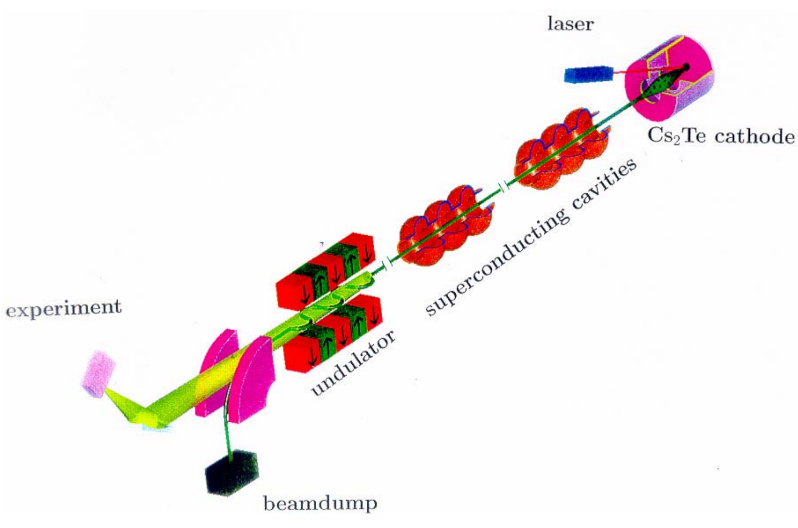

Fig. 1 Schematic of a linac-driven free-electron laser. Major components are i) a source of electron bunches of high charge density, ii) a linear accelerator (using superconducting technology is preferable to achieve a high duty cycle, but is not a must), iii) a long undulator magnet generating periodically alternating deflection of the electron beam, and iiii) a bending magnet separating the FEL radiation generated in the undulator from the electron beam.

1.2 Electron motion in the undulator field

In the present paper, we restrict ourselves to helical undulators, because this simplifies calculations. Extension to planar undulators can be found in the literature. It modifies some quantitative results but it doesn't change essentials.

In the vicinity of the axis of a helical undulator with period length λ_u , the magnetic field can be expressed (to first order in the distance r to the axis) by

$$\vec{\mathbf{B}} = \mathbf{B}\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \\ 0 \end{pmatrix} + O(r^2) \quad \left(\text{using } k_u = \frac{2\pi}{\lambda_u}\right) \tag{5}$$

The equation of motion of the electron in this field is

$$m\gamma\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \vec{\mathbf{B}} = q\mathbf{B}\begin{pmatrix} -\dot{z}\cdot\cos(k_u z) \\ -\dot{z}\cdot\sin(k_u z) \\ \dot{x}\cdot\cos(k_u z) + \dot{y}\cdot\sin(k_u z) \end{pmatrix} \tag{6}$$

One solution to this equation is a periodic, helical motion:

longitudinal motion: v_z = const., $z = v_z t = \beta_z ct$

transverse motion on a circle: $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c\frac{K}{\gamma}\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \end{pmatrix}$, or, using $w = x + iy$

$$\dot{w} = ic\frac{K}{\gamma}\exp(ik_u z) \tag{7}$$

This can be solved easily: $w = \dfrac{cK}{\gamma k_u v_z}\exp(ik_u z)$.

Here, $K = \dfrac{e\lambda_u \mathbf{B}}{2\pi m_0 c}$. K is called undulator parameter. It is typically $K \approx 1$

The opening angle of helical motion is seen to be $\dfrac{v_\perp}{c} = \dfrac{K}{\gamma} \approx \dfrac{1}{\gamma}$

With this result, we can now determine $\beta_z = \dfrac{v_z}{c}$:

$$\beta_z = \frac{1}{c}\sqrt{v^2 - \dot{x}^2 - \dot{y}^2} = \sqrt{\beta^2 - \left(\frac{K}{\gamma}\right)^2} = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{K}{\gamma}\right)^2} \approx 1 - \frac{1}{2\gamma^2}\left(1 + K^2\right) \tag{8}$$

1.3 Interaction with electromagnetic wave

We consider an external electromagnetic wave moving parallel to the electron beam, i.e. in z-direction. Let's assume a plane wave, which has zero z-component of the electric field vector:

$$\vec{\mathbf{E}}_L = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix}, \tag{9}$$

with the magnetic field given by: $\vec{\mathbf{B}}_L = \frac{1}{c\omega_L}\dot{\vec{\mathbf{E}}}_L$. ω_L is the angular frequency of the e.m. "light"wave and the index L stand for "light". It is certainly unnecessary mentioning that this frequency doesn't need to be at all in the visible range. $k_L = \frac{2\pi}{\lambda_L}$ is the wave number. Again, complex notation is very useful, because we have to deal with only two components of $\vec{\mathbf{E}}_L$. We define a complex electric field given by $\mathbf{E}_L = \mathbf{E}_{L,x} + i\mathbf{E}_{L,y} = \mathbf{E}_0 \exp i(\omega_L t - k_L z - \varphi_0)$.

We now calculate the change of the electron's energy in the combined presence of the undulator and the e.m. field. It is well known, that a charged particle doesn't gain energy in any magnetic field, since the Lorentz force is always perpendicular to the particle's velocity. Thus we have to consider the electric field only. As the e.m. wave has only electric field components perpendicular to the mean electron beam (z-)direction, we now recognize the important role of the undulator field: It generates velocity components of the electrons in the direction of the electric field vector, i.e. in the x- and y-direction and thus makes energy transfer between the e.m. wave and the electron beam possible. The electron's energy E is changed at a rate

$$\begin{aligned} \frac{dE}{dt} &= mc^2\frac{d\gamma}{dt} = \vec{v}\cdot\vec{F} = q\cdot\vec{v}\cdot\vec{\mathbf{E}}_L = q\left(\dot{x}\mathbf{E}_{L,x} + \dot{y}\mathbf{E}_{L,y}\right) = q\Re\left(\dot{w}\mathbf{E}_L^*\right) = \\ &= -qc\frac{K\mathbf{E}_0}{\gamma}\sin\left\{(k_u + k_L)z - \omega_L t + \varphi_0\right\} = -qc\frac{K\mathbf{E}_0}{\gamma}\sin\Psi \end{aligned}$$

Here we have used Eq. (7) and the "ponderomotive phase" defined as $\Psi = (k_u + k_L)z - \omega_L t + \varphi_0$. If we use $z = v_z t = \beta_z ct$, we can write

$$\boxed{\Psi = (k_u + k_L)z - \frac{\omega_L z}{\beta_z c} + \varphi_0} \tag{10}$$

and

$$\boxed{\frac{dE}{dz} = -\frac{q\mathbf{E}_0 K}{\gamma\beta_z}\sin\Psi}. \tag{11}$$

The energy dE is taken from or transferred to the radiation field. For most frequencies, dE/dt oscillates very rapidly. A significant energy transfer will only be accumulated if the phase difference between particle motion and e.m. wave stays constant with time. Thus, there is a resonance condition give by $\Psi = const. \rightarrow \quad \frac{d\Psi}{dz} = (k_u + k_L) - \frac{\omega_L}{\beta_z c} = 0$. Using $\omega_L = ck_L$ yields $k_u + k_L - \frac{k_L}{\beta_z} = 0$.

Solving for $\lambda_L = \dfrac{2\pi}{k_L}$ we get the resonance condition

$$\lambda_{\mathrm{L}} = \lambda_u \frac{1-\beta_z}{\beta_z} \approx \lambda_u \left(1-\beta_z\right) \approx \frac{\lambda_u}{2\gamma^2}\left(1+K^2\right) \tag{12}$$

It is important to realize that the resonant wavelength λ_L is identical to the on-axis, first harmonic wavelength spontaneously radiated by the undulator.

With Eq. (12) we have achieved a condition for continuous energy transfer from the electron beam to the e.m. wave. However, even if all electrons would have exactly the right energy to fulfill this condition when they enter the undulator, they will leave the resonance energy quickly due to the energy transfer to (or from) the wave. Thus, we need to investigate what happens to electrons particles with energies slightly off resonance. For particles slightly off resonance, the phase Ψ will slip. In order to understand by how much, we note that in Eq. (10) only $\beta_z \approx 1 - \dfrac{1}{2\gamma^2}\left(1+K^2\right)$ depends on energy. Writing $\gamma = \gamma_{\mathrm{res}} + \Delta\gamma$ we get

$$\begin{aligned} \frac{d\Psi}{dz} &= \left(k_u + k_L\right) - \frac{\omega_L}{c\left(1 - \dfrac{1+K^2}{2\left(\gamma_{\mathrm{res}}+\Delta\gamma\right)^2}\right)} \approx k_u + k_L - \frac{\omega_L}{\beta_z\left(\gamma_{res}\right)\cdot c} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{\mathrm{res}}{}^3}\Delta\gamma = \\ &= \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{\mathrm{res}}{}^3}\Delta\gamma = k_u \frac{2}{\gamma_{\mathrm{res}}}\Delta\gamma \end{aligned} \tag{13}$$

Deriving once more with respect to *z* yields $\dfrac{d^2\Psi}{dz^2} = k_u \dfrac{2}{\gamma_{\mathrm{res}}}\dfrac{d\gamma}{dz}$.

Using Eq. (11) in the form $\dfrac{d\gamma}{dz} = -\dfrac{q\mathbf{E}_0 K}{m_0 c^2 \gamma\beta_z}\,\mathbf{sin}\,\Psi$ we finally get

$$\boxed{\frac{d^2\Psi}{dz^2} = -\frac{2q}{m_0c^2}\frac{\mathbf{E}_0 Kk_u}{\gamma_{res}^2\beta_z}\,\mathbf{sin}\,\Psi = -\Omega^2\,\mathbf{sin}\,\Psi \quad \text{with} \quad \Omega^2 = \frac{2q}{m_0c^2}\frac{\mathbf{E}_0 Kk_u}{\gamma_{res}^2\beta_z}} \tag{14}$$

This is a pendulum equation in the $\Delta\gamma$ - Ψ phase space: electrons with little deviation from resonance energy or from synchronous phase perform periodic oscillations, see Fig. 2. This is equivalent to the synchrotron oscillations in storage rings, with the difference that the “bucket” length is now the optical wavelength. Like in synchrotron oscillation, particles within the separatrix get bunched.

The energy lost (or gained) by an electron increases (or decreases, respectively) the field energy. Thus, as seen from Eq. (11) and illustrated in Fig. 2, there is gain or loss in field energy per undulator passage depending on where the electron starts in the $\Delta\gamma$ - Ψ phase space.

1.3.1 The Separatrix

In order to determine the parameters of the separatrix, we look for a first integral of Eq. (14):

Multiplying $\Psi'' = -\Omega^2 \sin\Psi$ by $2\Psi'$ on both sides and using $2\Psi'\Psi'' = \frac{d}{dz}(\Psi')^2$ yields

$$\int 2\Psi'\Psi'' dz = (\Psi')^2 = 2\int -\Omega^2 \sin\Psi\Psi' dz = 2\int -\Omega^2 \sin\Psi d\Psi = 2\Omega^2 \cos\Psi + const$$

With $\Psi' = k_u \frac{2}{\gamma_{res}}\Delta\gamma$ this reads $\left(k_u \frac{2}{\gamma_{res}}\Delta\gamma\right)^2 = 2\Omega^2 \cos\Psi + const$, thus

$$\boxed{(\Delta\gamma)^2 - \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}\cos\Psi = const.} \tag{15}$$

with *const.* determined by initial conditions.

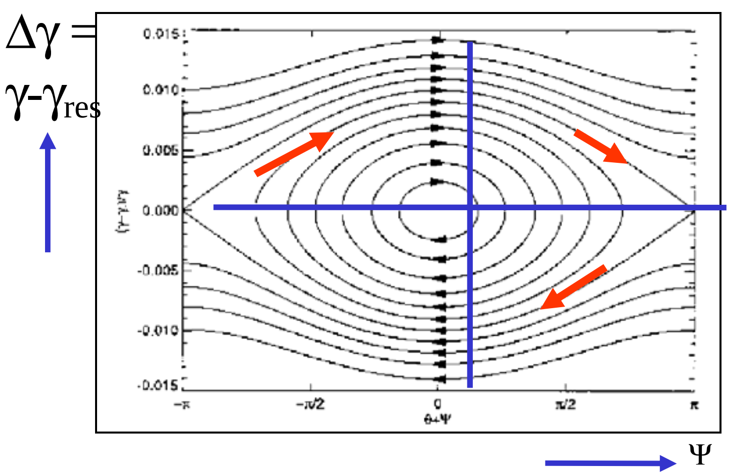

Fig. 2 In presence of the undulator field and the e.m. field, buckets are formed where electrons perform periodic oscillation, if the deviation from resonance energy and from synchronous phase is small. In contrast to synchrotron oscillation buckets, the longitudinal size of our buckets is very small, i. e. the optical wavelength.

There are two cases to be distinguished:

Case 1: $const. < \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. Then, $\Delta\gamma = \sqrt{const. + \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}\cos\Psi}$ has real solutions only within a limited range of phases. This is the case of rotation within the separatrix.

Case 2: $const. > \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. In this case all phases are possible, but $\Delta\gamma = 0$ cannot be reached. As a consequence, the electron performs “libration” outside the separatrix. The separatrix is defined by the limiting case: $const. = \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. Thus, the separatrix is defined by the equation

$$\boxed{(\Delta\gamma)^2 = \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}(1+\cos\Psi)}. \tag{16}$$

The height of the separatrix is given by: $\Delta\gamma_{\mathbf{max}} - \Delta\gamma_{\mathbf{min}} = \sqrt{\frac{2q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}}$, (17)

i.e. it is determined by the strengths of both the external e.m. wave $\mathbf{E}_0$ and of the undulator field (through *K*).

1.3.2 Power Gain

In practice, an electron beam consists of many particles distributed smoothly over all phases, so that it is not obvious from the previous analysis whether a significant over-all amplification of the e.m. wave can take place at all. We are now going to determine the "power gain" of the FEL in presence of the entire beam. Our most important assumption will be that the amplitude of the e.m. wave will change only little during one passage of the electron beam, i.e. the power gain (as defined below) is much smaller than unity: |G| < 1. This is the "low gain approximation" which is the subject of this chapter. We also assume an initially monoenergetic beam with some deviation $\Delta\gamma$ from resonance energy. Let's define the power gain G_i due to a particle identified by the index i by

$$\begin{aligned} G_i &= \frac{\text{gain of field energy produced by electron i}}{\text{total field energy}} = \frac{-mc^2\left(\gamma_i(z=L_u)-\gamma_i(0)\right)}{\frac{\varepsilon_0}{2}\mathbf{E}_0^2 \cdot V} \\ &= \frac{-mc^2\gamma_{\text{res}}\left(\Psi_i'(z=L_u)-\Psi_i'(0)\right)}{\varepsilon_0\,\mathbf{E}_0^2\cdot Vk_u} \text{ (using Eq. (13) } \Delta\gamma = \frac{\gamma_{\text{res}}}{2k_u}\Psi' \text{)} \end{aligned} \tag{18}$$

L_u is the length of the undulator. Calculation of G_i requires solution of the pendulum equation (14) $\Psi'' = -\Omega^2 \sin\Psi$ for $\Psi(z)$. This is done iteratively. We start with the

ansatz $$\Psi(z) = \Psi_0 + \Psi_0' \cdot z + \delta\Psi(z), \tag{19}$$

where $\delta\Psi(z)$ is the higher order term.

<u>Step 1:</u> $\delta\Psi(z) = 0$.

Using the ansatz, a first integral of Eq. (14) is then:

$$\Psi'^{(1)} = \Psi_0' - \Omega^2 \int_0^z \sin\left(\Psi_0 + \Psi_0' \cdot \tilde{z}\right) d\tilde{z} = \Psi_0' - \Omega^2 \frac{1}{\Psi_0'}\left[\cos\Psi_0 - \cos\left(\Psi_0 + \Psi_0' \cdot z\right)\right] \tag{20}$$

The gain of the entire beam (consisting of N_p particles) is given by

$$G = \sum_i G_i = \left\langle G_i \right\rangle_{\Psi_{i0}} \cdot N_p \tag{21}$$

Averaging with respect to the initial phases is denoted by $\left\langle \ \right\rangle_{\Psi_{i0}}$ and yields

$$\left\langle \Psi'^{(1)} - \Psi'_0 \right\rangle_{\Psi_0} = \left\langle -\Omega^2 \frac{1}{\Psi'_0}\left[\cos\Psi_0 - \cos\left(\Psi_0 + \Psi'_0 \cdot z\right)\right] \right\rangle_{\Psi_0} = 0 \tag{22}$$

The important result is that, in first order, the average gain *G* is zero!

<u>Step 2:</u> $\delta\Psi(z) \neq 0$, calculating $\delta\Psi(z)$ using the results of step 1, Eq. (20).

$$\begin{aligned} \delta\Psi(z) &= -\Omega^2 \frac{1}{\Psi'_0} \int_0^z \left[\cos\Psi_0 - \cos\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right)\right] d\tilde{z} = \\ &= -\Omega^2 \frac{1}{\Psi'_0}\left[z \cdot \cos\Psi_0 - \frac{1}{\Psi'_0}\sin\left(\Psi_0 + \Psi'_0 \cdot z\right) + \frac{1}{\Psi'_0}\sin\Psi_0 \right] \end{aligned} \tag{23}$$

Using again the ansatz, the first integral of Eq. (14) now reads

$$\Psi'^{(2)} - \Psi'_0 =$$

$$-\Omega^2 \int_0^z \sin\left(\Psi_0 + \Psi'_0 \cdot \tilde{z} + \delta\Psi(\tilde{z})\right) d\tilde{z} \approx -\Omega^2 \int_0^z \left[\sin\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right) + \cos\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right)\delta\Psi(\tilde{z})\right] d\tilde{z}$$

The approximation is valid for $\delta\Psi(\tilde{z}) \ll \pi$ which should be kept in mind when using the results. Plugging in $\delta\Psi(\tilde{z})$ and averaging with respect to phases yields

$$\begin{aligned} \left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} &= \Omega^4 \frac{1}{\Psi_0'^2} \left\langle \int_0^{Lu} \left[\tilde{z}\Psi'_0 \cos^2\Psi_0 \cdot \cos\left(\Psi'_0 \cdot \tilde{z}\right) - \sin^2\Psi_0 \cdot \sin\left(\Psi'_0 \cdot \tilde{z}\right)\right] d\tilde{z} \right\rangle_{\Psi_0} = \\ &= \frac{\Omega^4}{2\Psi_0'^2} \int_0^{Lu} \left[\tilde{z}\Psi'_0 \cos\left(\Psi'_0 \cdot \tilde{z}\right) - \sin\left(\Psi'_0 \cdot \tilde{z}\right)\right] d\tilde{z} \end{aligned}$$

Here we have used $\left\langle \cos(\alpha+\beta)\sin(\alpha+\beta) \right\rangle_\alpha = 0$ and $\left\langle \cos^2\alpha \right\rangle_\alpha = \frac{1}{2}$; and $\left\langle \sin^2\alpha \right\rangle_\alpha = \frac{1}{2}$. By partial integration[1] we get

[1] Explicitly, the partial integration reads

$$\int_0^{L_u} \underbrace{\tilde{z}\Psi'_0}_{u} \underbrace{\cos}_{v'}(\Psi'_0\tilde{z})d\tilde{z} = \underbrace{\tilde{z}\Psi'_0}_{u} \underbrace{\frac{1}{\Psi'_0}\sin(\tilde{z}\Psi'_0)}_{v}\Big|_0^{L_u} - \int_0^{L_u} \underbrace{\Psi'_0}_{u'} \underbrace{\frac{1}{\Psi'_0}\sin}_{v}(\Psi'_0\tilde{z})d\tilde{z}$$

$$\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \frac{\Omega^4}{2\Psi'^2_0}\left[L_u \sin\left(\Psi'_0 L_u\right) - 2\int_0^{Lu} \sin\left(\Psi'_0 \cdot \tilde{z}\right) d\tilde{z} \right] =$$

$$= \frac{\Omega^4}{2\Psi'^2_0}\left[L_u \sin\left(\Psi'_0 L_u\right) + 2\frac{1}{\Psi'_0}\left(\cos\left(\Psi'_0 L_u\right) - 1\right) \right] =$$

(using $\sin 2x = 2\sin x \cos x$ and $\cos 2x - 1 = -2\sin^2 x$)

$$= \Omega^4 \left(\frac{1}{\Psi'^2_0} L_u \sin\frac{\Psi'_0 L_u}{2} \cos\frac{\Psi'_0 L_u}{2} - \frac{2}{\Psi'^3_0} \sin^2\frac{\Psi'_0 L_u}{2} \right)$$

The latter expression can also be put in the form $\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \Omega^4 \frac{\mathrm{d}}{\mathrm{d}\Psi'_0}\left\{ \frac{\sin^2 \frac{\Psi'_0 L_u}{2}}{\Psi'^2_0} \right\}$

With Eqs. (18) and (21), the gain can we written

$$G = N_p \frac{-mc^2\gamma_{\mathrm{res}}\left(\Psi'_i\left(z = L_u\right) - \Psi'_i(0)\right)}{\varepsilon_0 \mathbf{E}_0^2 \cdot V k_u} = -\frac{mc^2\gamma_{res} N_p}{\varepsilon_0 \mathbf{E}_0^2 \cdot V k_u}\Omega^4 \frac{\mathrm{d}}{\mathrm{d}\Psi'_0}\left\{ \frac{\sin^2 \frac{\Psi'_0 \cdot L_u}{2}}{\Psi'^2_0} \right\}$$

We now use the abbreviations $\xi = \frac{\Psi'_0 \cdot L_u}{2}$, $n_p = \frac{N_p}{V}$, $L_u = N_u \lambda_u$ (with N_u the number of undulator periods) and get the final result [2]

$$\boxed{G = -\frac{\pi q^2 N_u^3 \lambda_u^2 K^2 n_p}{\varepsilon_0 mc^2 \gamma^3} \frac{\mathrm{d}}{\mathrm{d}\xi}\left\{ \frac{\sin^2 \xi}{\xi^2} \right\}} \tag{24}$$

The functions $\frac{\sin^2 \xi}{\xi^2}$ and $\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{ \frac{\sin^2 \xi}{\xi^2} \right\}$ are plotted in Fig. 3.

Let us summarize the key assumptions made for this result: a helical undulator, perfect overlap of electron and radiation field, perfect electron beam (zero emittance and no momentum spread). It is noted that the approximations don't assume that the beam is located inside the separatrix, i.e. the external e.m. field may be so weak that the separatrix covers only a small fraction of the gain curve.

For the interpretation of Eq. (24) it is useful to express ξ in the form

$$\xi = \frac{\Psi'_0 \cdot L_u}{2} = \frac{k_u L_u \Delta\gamma}{\gamma_{res}} = 2\pi N_u \frac{\Delta\gamma}{\gamma_{res}} \tag{25}$$

[2] Using the same Volume V in the definition of the particle density as in the expression for the total field energy (see Eq. (18)) means that we assume perfect overlap of electron beam and e.m. field.

$$\frac{\sin^2\xi}{\xi^2} \qquad\qquad -\frac{d}{d\xi}\frac{\sin^2\xi}{\xi^2} = \frac{1}{\xi^3}\left(1-\cos(2\xi)-\xi\sin(2\xi)\right)$$

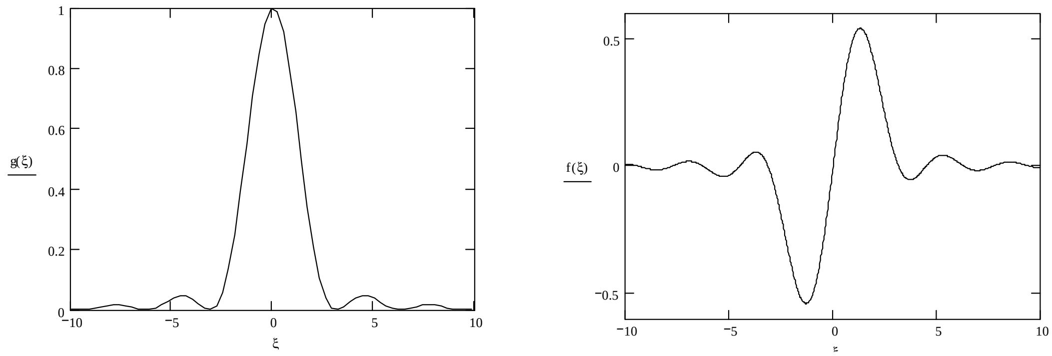

Fig. 3 In low-gain approximation, the dependency of power gain on the initial momentum (right) can be written as the derivative of the line shape function (left) of the spontaneous undulator radiation.

Thus, we can interpret the result as follows: To first order in the iteration, there is no net gain (G=0), because phase space motion is (almost) symmetric: As many particles move up as down. In second order it is seen however that, for positive $\Delta\gamma$, the motion of particles with positive phase goes more rapidly downwards than the motion of the others goes upwards, i.e. there is positive gain if the electron energy is slightly above resonance energy ($\Delta\gamma$>0). This is illustrated schematically in Fig. 4. There is no gain for particles precisely on resonance energy ($\Delta\gamma$=0). For $\Delta\gamma$<0, the gain is negative, i.e. the beam extracts energy from the e.m. wave, i.e. it gets accelerated. A device accelerating electrons by an e.m. wave using this mechanism is called an “inverse FEL”.

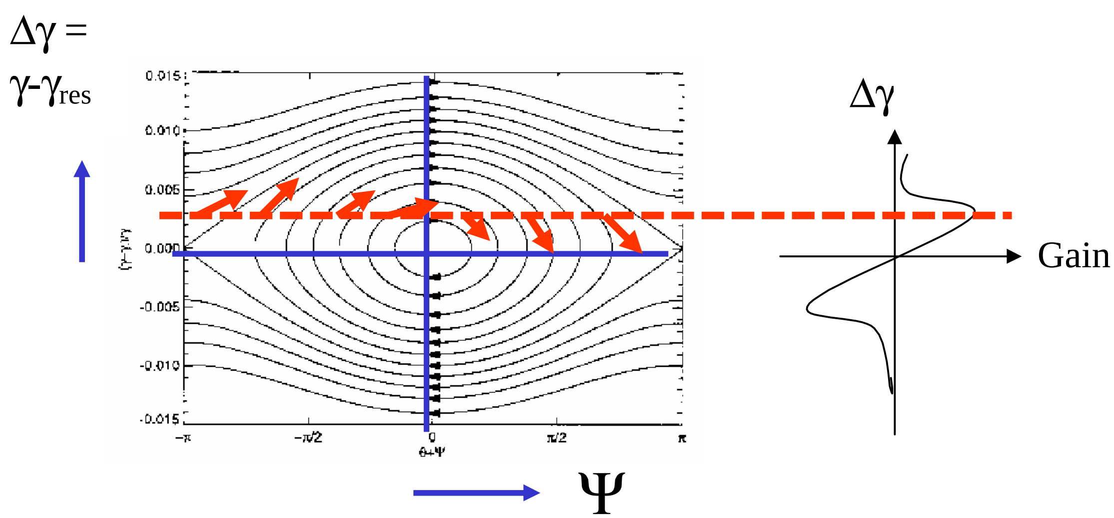

Fig. 4 If the electron energy is slightly above resonance energy (red broken line), some particles lose, some gain energy. In average, electron energy is pumped into the e.m. wave (positive power gain), which is a second order effect.

1.3.3 Madey Theorem

Another interpretation of Eq. (24) makes use of the relation (see Eq. (12)) $\frac{\Delta\omega}{2\omega_{\mathrm{res}}}=\frac{\Delta\gamma}{\gamma_{res}}$ connecting $\Delta\gamma$ to the deviation of the angular frequency $\Delta\omega$ from its resonance value ω_{res}. Using this relation, we can write $G=-\frac{4\pi q^2 N_u^2 \lambda_u n_p}{\varepsilon_0 mc\gamma}\frac{K^2}{(1+K^2)}\frac{d}{d\omega}\frac{\sin^2\left(\pi N_u \frac{\Delta\omega}{\omega_{\mathrm{res}}}\right)}{\left(\pi N_u \frac{\Delta\omega}{\omega_{\mathrm{res}}}\right)^2}$. The expression $\frac{\sin^2\left(\pi N_u \frac{\Delta\omega}{\omega_{\mathrm{res}}}\right)}{\left(\pi N_u \frac{\Delta\omega}{\omega_{\mathrm{res}}}\right)^2}$ is just the spectral line shape function of spontaneous radiation of an undulator with N_u periods in the vicinity of the first harmonic resonance frequency ω_{res}, see the left hand side of Fig. 3. Thus, the following statement can be made, known as the **Madey-Theorem** [5]:

The gain function of low gain FEL emission is the derivative of the line shape function of spontaneous undulator radiation.

1.3.4 The Optical Cavity

The amount of radiation energy produced per undulator passage is $\Delta E = G\cdot E_i$, with E_i the radiation energy before the electron bunch has passed the undulator. One might think that, for applications, a few % power gain (i.e. a low gain FEL) is of no interest. However, it is important to realize that the gain is independent of the strength of the initial, external e.m. field, i.e. whatever the initial field is, it will be amplified by this gain factor. With a pair of mirrors, arranged to form an optical cavity as shown in Fig. 5, one can accumulate the produced radiation successively, if on each round trip of radiation there is a fresh electron bunch available. After N round trips, the total power gain is $G_{total}=G^N$, which may be a very large number indeed, even if G is not much larger than unity.

At the end of this process, there is very much radiation energy stored in the optical cavity, so that even few percent amplification of this energy is a large quantity in terms of absolute numbers. In other words, the electrons are stimulated to emit radiation due to the presence of the existing field. In fact, the amplification process in the FEL can be described quantum-mechanically in terms of emission and absorption of radiation quanta (photons), which justifies - together with the properties of FEL radiation like coherence and many photons per coherence volume - the notion of a “laser”. Many early papers on FELs were done in the framework of quantum mechanics which explains the quantum-based terminology widely used in the FEL community. Nevertheless, the description of FELs in terms of classical physics is perfectly correct, with very few exceptions only relevant in rare cases. The FEL is a “classical device”.

Some fraction of the gained radiation energy is extracted through one of the mirrors which is made semi-transparent. Of course, the mirror transparency must be arranged such that the total power losses of the optical cavity by extraction or absorption don’t exceed the power gain.

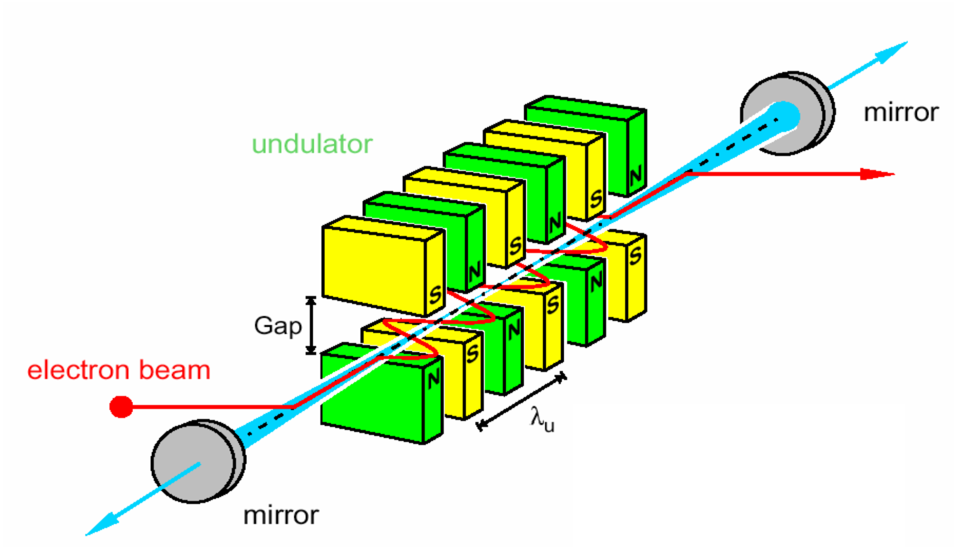

Fig. 5 Schematic of a free-electron laser in oscillator mode (courtesy: R. Bakker). Even if there is only a few percent field gain per passage of an electron bunch (low-gain FEL), large radiation power can be generated if the radiation field can be stored in an optical cavity and if many electron bunches pass the cavity at a timing synchronized with the round-trip of radiation within the oscillator. If the power gain exceeds the accumulated mirror losses (including the semi-transparent mirror for extraction), the stored power increases passage by passage in an exponential way until saturation is reached.

1.3.5 Saturation

If the change of electron energy within one undulator passage becomes comparable with $\frac{\gamma_{res}}{2\pi N_u}$, the phase advance per undulator passage becomes large according to Eq. (13), such that the assumption $\delta\Psi(\tilde{z}) \ll \pi$ made for our gain calculation is violated. Also, the assumption of a quasi-monoenergetic electron beam is violated, i.e. the parameter ξ defined in Eq. (25) varies significantly during the undulator passage, depending on the longitudinal position within the separatrix. Thus, Eq. (24) – anyhow not accurate any more since based on violated assumptions – becomes useless to calculate the over-all gain. In this case, the gain must be calculated numerically.

According to Eq. (17), the height of the separatrix grows with the radiation energy stored in the cavity. Thus, more and more electrons get trapped within the separatrix, providing an efficient mechanism for longitudinal bunching of the electron beam at the optical wavelength (an effect sometimes called "micro-bunching", to be distinguished from longitudinal bunching of the entire electron bunch). It turns out that the gain process saturates if most of the electrons get micro-bunched, i.e. if the electron density is almost completely modulated at the resonance wavelength. According to Eq. (4) and the explanations given there, in this case the undulator radiation power exceeds spontaneous radiation power by a large factor N comparable to the number of electrons per resonance wavelength.

In the present chapter, only the kinematic problem of the electron beam in combined presence of a given e.m. wave and an undulator field has been solved, while the gain of radiation power was derived from an energy conservation argument: The kinetic energy taken from the electron must go into the radiation energy – where else? It is obvious, that for a more thorough analysis, one has to solve both the kinematic problem *and* the electro-dynamical problem (i.e. the generation of radiation by a modulated charge according to Maxwell's equations) simultaneously. Such a detailed analysis of the FEL oscillator, including the analysis of saturation, can be found in the literature, e.g. Ref. [2]. In the next chapter we will present a treatment of this kind for the high-gain, single-pass FEL. Indeed, the low-gain results of the present chapter can be derived from the general result as a special case, namely the low-gain approximation, which justifies with hindsight the low-gain treatment given above.

Although storage-ring FELs are beyond the scope of this article, let's conclude this section with a remark on FEL oscillators driven by electron bunches in a storage ring. In principle, such an arrangement is very attractive, since the electron bunch can be used many times (i.e. once per revolution), and reliable operation can be expected due to the inherent stability of storage rings in terms of timing and bunch population. However, there are also inherent drawbacks: As the same electron bunch is used many times, the electron beam dynamics in the storage ring must be taken into account. As described above, close to saturation the energy width of the electron beam is increased considerably at each passage of the FEL. This energy broadening accumulates from turn to turn. It is to some extent compensated by radiation damping, but the saturation process remains drastically determined by this effect [6]. But even if one would consider using the beam of a storage ring only once per damping time, there is a fundamental issue which makes such an electron bunch unattractive for some cases: The product of bunch length and energy width (the so-called longitudinal emittance) is determined by quantum fluctuation effects in a storage ring and cannot be made as small as in a linear accelerator.

1.3.6 Start-up from noise

In order to achieve maximum gain, ξ should be ~ +1, i.e. $\Delta\gamma \approx + \gamma/2\pi N_u$. Thus, the electron beam energy should be above resonance energy by that amount. This is easy to achieve if the initial e-m field is provided by an external source: The external wavelength determines, together with the undulator parameters, the resonance energy γ_{res} of the electron beam. Therefore, we just have to set the electron beam energy to $\gamma_{res} + \Delta\gamma$.

But what happens, if there is no external radiation source? The FEL can still work, if the spontaneous radiation of the undulator is used. Using two mirrors, this radiation has to be reflected back to the entrance of the undulator, and it must be synchronized longitudinally with the next electron bunch for overlap in the undulator (see Fig. 5).

Unfortunately, if we want to use the center of the spectrum of the spontaneous undulator radiation as the "external wave", there will be no FEL gain, because this wavelength, together with the beam energy, exactly fulfills the resonance condition. What helps is, that the spontaneous spectrum has the same width (see Eq. (24) and Fig. 3) as the gain curve, thus there is always significant power at a wavelength with high gain.

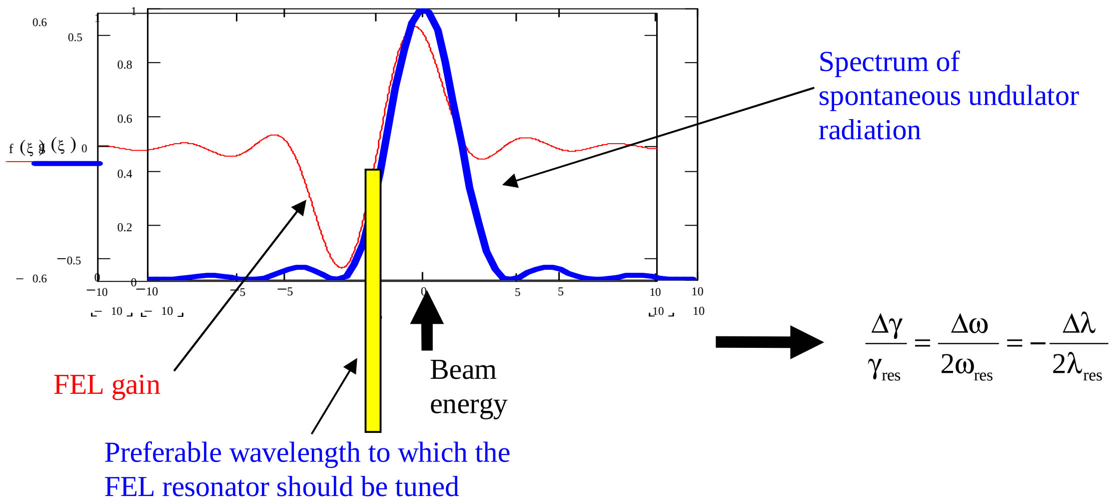

Fig. 6 When starting from noise, the spontaneous radiation of the undulator serves as "external e.m. wave" to be amplified in the FEL. For optimum gain, the electron energy should be chosen such that it samples the gain curve where it is positive and maximum. Since the gain curve is centered with respect to the resonance energy (rather than w.r.t. the beam energy), this condition can be met, if the resonant wavelength of the optical cavity (yellow line) is tuned below the electron beam energy. In this way, only the low-frequency wing of the undulator spectrum gets amplified, but it is guaranteed by the strict relation between width of the spectrum and width of the gain curve (see section 1.3.3) that there is sufficient radiation power in this portion of the spectrum to serve as input signal.

This can happen in two ways:

1. If the bandwidth of the optical resonator formed by the two mirrors is very large, then the FEL will "automatically" amplify only that part of the spectrum with positive gain. This will happen with the lower frequency part of the spontaneous spectrum, since for this wavelength to be resonant, the beam energy would have to be smaller than it actually is, so the actual beam energy will be slightly above resonance, as it should be for positive gain.
2. If the bandwidth of the optical resonator is small (normal case), it should be tuned below the center frequency of the spontaneous spectrum (same argument as before). This is illustrated in Fig. 6.

2. THE HIGH-GAIN FREE-ELECTRON LASER

If the radiation power gained within a single passage of the electron beam through the undulator is comparable or much larger than the input radiation power, the low-gain approximation is not applicable any more. In this case, we have to take into account the time-dependence of the increasing electro-magnetic wave as determined by the motion of the electrons in the beam. On the other hand, just this motion is determined by the e.m. field amplitude at any point in time and space. Thus, we need to treat the evolution of electron kinematics and e.m. field amplitude in a self-consistent manner.

It is the purpose of this chapter to derive the key equations from first principles, motivating the approximations and providing some realistic numbers for illustration. The treatment follows closely the one given in Ref. [2].

2.1 Generation of electro-magnetic fields by the electron beam

We start with a derivation of the wave equation for the electric field from Maxwell's equation. From Maxwell equation $\mathrm{rot}\,\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ we get $\mathrm{rot}\,\mathrm{rot}\,\vec{\mathbf{E}} = \mathrm{grad}\,\mathrm{div}\vec{\mathbf{E}} - \nabla^2 \vec{\mathbf{E}} = -\mu_0 \mathrm{rot}\frac{\partial \vec{\mathbf{H}}}{\partial t}$. (26)

Next, we derive Maxwell equation $\mathrm{rot}\,\vec{\mathbf{H}} = \vec{j} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ once more with respect to time and get $\mathrm{rot}\frac{\partial \vec{\mathbf{H}}}{\partial t} = \frac{\partial \vec{j}}{\partial t} + \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$.

If we further use Maxwell equations $\varepsilon_0 \mathrm{div}\vec{\mathbf{E}} = \rho$, Eq. (26) can be written in the form

$\frac{1}{\varepsilon_0} grad\rho - \nabla^2 \vec{\mathbf{E}} = -\mu_0 \frac{\partial \vec{j}}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$. Using $\mu_0 \varepsilon_0 = \frac{1}{c^2}$, this reads

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2} \right) \vec{\mathbf{E}} = \mu_0 \frac{\partial \vec{j}}{\partial t} + \frac{1}{\varepsilon_0} \nabla \rho \,, \qquad (27)$$

which is the well-known wave equation for the electric field. In the following, we will restrict ourselves to a one-dimensional treatment of the FEL, i.e. we consider a purely transverse e.m. field. This means, in particular, that we neglect diffraction effects, which is certainly questionable for long wavelengths. With this approximation, Eq. (27) reads $\frac{\partial^2 \mathbf{E}_\perp}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_\perp}{\partial t^2} = \mu_0 \frac{\partial j_\perp}{\partial t} + \frac{1}{\varepsilon_0} \nabla_\perp \rho$, (28)

with the index $\perp$ denoting the vector component perpendicular to the direction of electron propagation.

The term $\frac{1}{\varepsilon_0} \nabla_\perp \rho$ in Eq. (28) can be neglected, because its contribution to radiation generation is small in all practical cases[3]. The transverse electric field $\mathbf{E}_\perp$ of the e.m. wave can be written in the form

$\vec{\mathbf{E}}_\perp = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix}$. The magnetic field of the e.m. wave is then $\vec{\mathbf{B}}_\perp = \frac{1}{c\omega_L}\dot{\vec{\mathbf{E}}}_\perp$.

In analogy to the previous chapter, we make profit of the fact that we have to deal with only two components of $\vec{\mathbf{E}}$ and define a complex electric field given by $\mathbf{E} = \mathbf{E}_{\perp,x} + i\,\mathbf{E}_{\perp,y} = \mathbf{E}_0 \,\mathbf{exp}\, i(\omega_L t - k_L z - \varphi_0)$. The only difference to the low-gain case is now, that the amplitude $\mathbf{E}_0$ and the phase φ_0 (which we will call ψ_E now) may vary with *z* (though slowly compared to $\omega_L t$). We thus separate the slow part from the rapidly oscillating part by writing: $\mathbf{E} = \mathbf{E}_0(z)\,\mathbf{exp}\, i(\omega_L t - k_L z - \psi_E) = \tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\, i(\omega_L t - k_L z)$, with the slow part $\tilde{\mathbf{E}}_0(z) = \mathbf{E}_0(z)\,\mathbf{exp}\, i\psi_E$ and $\tilde{\mathbf{E}}_0^*(z)$ its complex conjugate (c.c.). Eq. (28), re-written for $\mathbf{E}_{\perp,x} + i\,\mathbf{E}_{\perp,y} = \tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\, i(\omega_L t - k_L z)$ reads

$$
\begin{aligned}
&\frac{\partial^2(\mathbf{E}_{\perp,x} + i\,\mathbf{E}_{\perp,y})}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2(\mathbf{E}_{\perp,x} + i\,\mathbf{E}_{\perp,y})}{\partial t^2} = \\
&= \frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\, i(\omega_L t - k_L z) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\, i(\omega_L t - k_L z) = \\
&\frac{\partial}{\partial z}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial z}\mathbf{exp}\, i(\omega_L t - k_L z) + \mathbf{exp}\, i(\omega_L t - k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right] - \ldots \\
&\ldots\frac{1}{c^2}\frac{\partial}{\partial t}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial t}\mathbf{exp}\, i(\omega_L t - k_L z) + \mathbf{exp}\, i(\omega_L t - k_L z)\cancel{\frac{\partial}{\partial t}\tilde{\mathbf{E}}_0^*(z)}\right] = \\
&\frac{\partial}{\partial z}(\tilde{\mathbf{E}}_0^*(z))\cdot(-ik_L)\,\mathbf{exp}\, i(\omega_L t - k_L z) + \tilde{\mathbf{E}}_0^*(z)(-k_L^2)\,\mathbf{exp}\, i(\omega_L t - k_L z) + \ldots \\
&\ldots(-ik_L)\,\mathbf{exp}\, i(\omega_L t - k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z) + \mathbf{exp}\, i(\omega_L t - k_L z)\frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z) - \ldots \\
&\cancel{\ldots\frac{1}{c^2}\frac{\partial}{\partial t}(\tilde{\mathbf{E}}_0^*(z))\cdot i\omega_L\,\mathbf{exp}\, i(\omega_L t - k_L z)} - \frac{1}{c^2}\tilde{\mathbf{E}}_0^*(z)(-\omega_L^2)\,\mathbf{exp}\, i(\omega_L t - k_L z) + 0
\end{aligned}
$$

where we have made use of our assumption that the complex field amplitude does depend on the longitudinal coordinate *z*, but not (explicitly) on time *t*, i.e. $\frac{\partial}{\partial t}\tilde{\mathbf{E}}_0^*(z) = 0$. We further neglect the second derivative of the field amplitude with respect to z, because it is assumed to vary only slowly and get:

[3] For a more detailed justification, see Ref. [2], chapter 4.1

$$\frac{\partial^2(\mathbf{E}_{\perp,x}+\mathrm{i}\,\mathbf{E}_{\perp,y})}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2(\mathbf{E}_{\perp,x}+\mathrm{i}\,\mathbf{E}_{\perp,y})}{\partial t^2}=$$
$$=-\left[2\mathrm{i}k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)-k_L^2\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)-\frac{(-\omega_L^2)}{c^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)=$$
$$=-\left[2\mathrm{i}k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)=\mu_0\frac{\partial(j_{\perp,x}+\mathrm{i}\cdot j_{\perp,y})}{\partial t}=\mathrm{i}\mu_0\frac{\mathrm{K}}{\gamma}\mathbf{exp}(\mathrm{i}k_u z)\,\frac{\partial j_z}{\partial t} \tag{29}$$

Here we made use of $\frac{\omega_L}{c}=k_L$. Also, we were able to relate the transverse components of the current density to its longitudinal component, since we know from Eq. (7) how electrons move in the presence of the helical undulator: Namely, because of $\vec{j}=r\vec{v}$, we were able to write $(\mathrm{j}_\mathrm{x}+i\cdot j_y)=(v_\mathrm{x}+i\cdot v_y)\frac{j_z}{v_z}=$ (see Eq.(7)) $=\mathrm{i}\frac{c}{v_z}\frac{K}{\gamma}e\,\mathbf{xp}(ik_u z)\,j_z\approx\mathrm{i}\frac{K}{\gamma}e\,\mathbf{xp}(ik_u z)\,j_z$.

Collecting the rapidly oscillating term and using $\Psi=(k_u+k_L)z-\omega_L t$, Eq. (29) re-writes:

$$\boxed{-\left[2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]=\mu_0\frac{\mathrm{K}}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,\mathrm{i}(k_u z+k_L z-\omega_L t)=\mu_0\frac{\mathrm{K}}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,\mathrm{i}\Psi} \tag{30}$$

The message of the equation is pretty simple: The electro-magnetic field amplitude is generated by the time-dependent current density.

To proceed further, we have to say something about the current density j_z.

2.2 Kinematics of electrons in phase space

j_z is determined by the initial charge distribution and its evolution in presence of the e.m. field and the undulator field. We know that electron dynamics is governed by the Hamiltonian

$$H(p_z,z,t)=\left[\left(p_z c-qA_z\right)^2+q^2\left(A_\perp+A_u\right)^2+m^2c^4\right]^{1/2}+q\phi\,,$$

with A_u describing the undulator field, and A_z, ϕ the space charge. Applying a canonical transformation, we can change from the canonical pair of coordinates z/p_z to Ψ/γ (actually, the pair is $\frac{\omega_\perp}{m_0c^2}\cdot\Psi/\gamma$, but $\frac{\omega_\perp}{m_0c^2}$ is constant), with $\Psi=(k_u+k_L)z-\omega_L t$ and γm_0c^2 the kinetic energy of the electron. A consequence of Hamiltonian mechanics is Liouville's Theorem, stating that phase space density *f* along the particle's motion is constant. Phase space motion must be described in any pair of canonically conjugate variables, and we choose Ψ/γ. In coordinates *z*, γ,Ψ, this theorem reads[4]:

$$\frac{df}{dz}=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial\Psi}\frac{\partial\Psi}{\partial z}+\frac{\partial f}{\partial\gamma}\frac{\partial\gamma}{\partial z}=0\text{, which is also called “Vlasov equation”} \tag{31}$$

[4] We use the longitudinal coordinate *z* as independent variable instead of time *t*, which in fact means another canonical transformation.

We have seen from Eq. (11), how the electron energy changes in the presence of e.m. field and undulator field. In addition to Eq. (11), we now include the energy gain due to the presence of a longitudinal space charge field $\mathbf{E}_z$: $\frac{d\gamma}{dz} = -\frac{q\mathbf{E}_0 K}{m_0 c^2 \gamma_0 \beta_z} \sin(\Psi + \psi_E) + \frac{q\mathbf{E}_z}{m_0 c^2}$. Of course, we have to understand that the electrical field strength $\mathbf{E}_0$ isn't constant any more. Therefore, we also allow for a slowly varying phase of the e.m. field, described by ψ_E .

$\frac{d\Psi}{dz}$ can be determined from Eq. (13): $\frac{d\Psi}{dz} = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma_0) \cdot c} + \frac{\omega_L}{c} \frac{1+K^2}{\gamma_0^3} \Delta\gamma$, where $\Delta\gamma$ denotes the deviation from γ_0. For the sake of generality, we allow now γ_0 to deviate slightly from resonance energy γ_{res} described by the detuning parameter $C(\gamma) = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma) \cdot c}$, i.e. $C(\gamma_{res}) = 0$. (Note: You may ask here, why the deviation from resonance energy is split into two terms, γ_0 and $\Delta\gamma$. The reason will become clear below, when we will use $\Delta\gamma$ to describe the energy distribution of the beam around the center γ_0). We get: $\frac{d\Psi}{dz} = C + \frac{\omega_L}{c} \frac{1+K^2}{\gamma_0^3} \Delta\gamma$. Eq. (31) now reads

$$\frac{\partial f}{\partial z} + \left(C + \frac{\omega_L}{c} \frac{1+K^2}{\gamma_0^3} \Delta\gamma \right) \frac{\partial f}{\partial \Psi} + \left(-\frac{qE_0 K}{m_0 c^2 \gamma_0} \sin(\Psi + \psi_E) + \frac{qE_z}{m_0 c^2} \right) \frac{\partial f}{\partial \gamma} = 0 . \tag{32}$$

Note that from now on we use $\beta_z \approx 1$. For the phase space density f we make the Ansatz $f(z, \gamma, \Psi) = f_0(\gamma) + f_1(z, \gamma) \cos(\Psi + \psi_0)$, i.e. we assume a density modulation at the optical wavelength, growing with z (in a way to be calculated), see Fig. 7 for illustration. The phase of this modulation is allowed to slowly depart from Ψ by ψ_0 (which is, in general different from ψ_E). In complex notation:

$f_1(z, \gamma) \cos(\Psi + \psi_0) = \frac{f_1}{2} e^{i(\Psi + \psi_0)} + \frac{f_1}{2} e^{-i(\Psi + \psi_0)} = \frac{f_1}{2} e^{i\psi_0} e^{i\Psi} + c.c. = \tilde{f}_1(z, \gamma) e^{i\Psi} + c.c.$ The complex amplitude $\tilde{f}_1(z, \gamma) = \frac{f_1}{2} e^{i\psi_0}$ of density modulation contains the slowly varying phase ψ_0. A similar Ansatz is made for the space charge field $\mathbf{E}_z$:

$\mathbf{E}_z = \mathrm{E}_z(z) \cos(\Psi + \psi_s) = \tilde{\mathbf{E}}_z(z) e^{i\Psi} + \mathrm{c.c.}$, again with its own slowly varying phase ψ_s .

Vlasov equation (32) can now be written:

$$\frac{\partial f}{\partial z}+\left(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\frac{\partial f}{\partial \Psi}+\left(-\frac{q\mathbf{E}_0\,K}{m_0c^2\gamma_0}\mathbf{sin}(\Psi+\psi_E)+\frac{q\mathbf{E}_z}{m_0c^2}\right)\frac{\partial f}{\partial \gamma}=$$

$$=\frac{\partial \tilde{f}_1}{\partial z}e^{i\Psi}+\frac{\partial \tilde{f}_1^*}{\partial z}e^{-i\Psi}+\left(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\left(i\tilde{f}_1e^{i\Psi}-i\tilde{f}_1^*e^{-i\Psi}\right)+$$

$$+\left[\frac{q\mathbf{E}_0\,K}{m_0c^2\gamma_0}\frac{i}{2}\left(e^{i(\Psi+\psi_E)}-e^{-i(\Psi+\psi_E)}\right)+\frac{q}{m_0c^2}\left(\tilde{\mathbf{E}}_z\,e^{i\Psi}+\tilde{\mathbf{E}}_z^*\,e^{-i\Psi}\right)\right]\left(\frac{\partial f_0}{\partial \gamma}+\frac{\partial \tilde{f}_1}{\partial \gamma}e^{i\Psi}+\frac{\partial \tilde{f}_1^*}{\partial \gamma}e^{-i\Psi}\right)=$$

(using $\tilde{\mathbf{E}}_0=\mathbf{E}_0\,e^{i\psi_E}$)

$$=e^{i\Psi}\left\{\begin{array}{l}\dfrac{\partial \tilde{f}_1}{\partial z}+i\left(C+\dfrac{\omega_L}{c}\dfrac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde{f}_1+\left(i\dfrac{q\tilde{\mathbf{E}}_0\,K}{2m_0c^2\gamma_0}+\dfrac{q}{m_0c^2}\tilde{\mathbf{E}}_z\right)\dfrac{\partial f_0}{\partial \gamma}+\\ \left[i\dfrac{q\mathbf{E}_0\,K}{2m_0c^2\gamma_0}\left(e^{i(\Psi+\psi_E)}-e^{-i(\Psi+\psi_E)}\right)+\dfrac{q}{m_0c^2}\left(\tilde{\mathbf{E}}_z\,e^{i\Psi}+\tilde{\mathbf{E}}_z^*\,e^{-i\Psi}\right)\right]\dfrac{\partial \tilde{f}_1}{\partial \gamma}\end{array}\right\}+c.c.=0$$

For this equation to hold for all phases Ψ, the expression in brackets $\{\ \}$ must vanish. Our next step in approximation assumes that the modulation amplitude doesn't depend on energy (see Fig. 7): $\frac{\partial \tilde{f}_1}{\partial \gamma}=0$. Then:

$$\boxed{\frac{\partial \tilde{f}_1(z,\gamma)}{\partial z}+i\left(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde{f}_1(z,\gamma)+\left(i\frac{q\tilde{\mathbf{E}}_0\,K}{2m_0c^2\gamma_0}+\frac{q}{m_0c^2}\tilde{\mathbf{E}}_z\right)\frac{\partial f_0(\gamma)}{\partial \gamma}=0}\cdot \qquad (33)$$

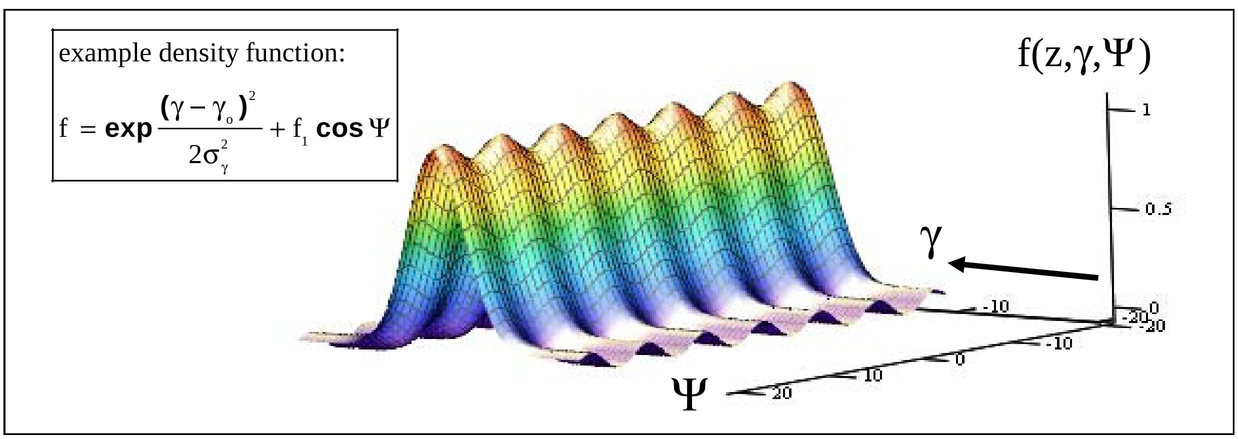

Fig. 7: Illustration of a possible phase space density function fulfilling the assumptions made here: The density modulation amplitude f_1 observed at an arbitrary location z doesn't depend on energy γ, and the amount of modulation in the core of the beam is small compared to the total density.

Eq. (33) is a differential equation in z of the type $\frac{df(z)}{dz} + i\alpha f(z) = g(z)$, which is solved by $f(z) = \int_0^z g(z') \exp[i\alpha(z'-z)]\, dz'$. Thus:

$$\tilde{f}_1(z,\gamma) = \int_0^z dz' \left[i \frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z') \right] \frac{\partial f_0(\gamma)}{\partial\gamma} \exp\left[i(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z) \right] \tag{34}$$

and $\tilde{f}_1^*(z,\gamma) = c.c.$ We can now calculate the current density:

$$j_z = \rho v_z \approx \rho c = qc\int f(z,\gamma,\Psi)d\gamma = qc\int f_0(\gamma)d\gamma + e^{i\Psi}qc\int \tilde{f}_1(z,\gamma)d\gamma + e^{-i\Psi}qc\int \tilde{f}_1^*(z,\gamma)d\gamma =$$
$$= j_0 + \tilde{j}_1 e^{i\Psi} + \tilde{j}_1^* e^{-i\Psi} \quad \text{,with} \quad \tilde{j}_1 = qc\int \tilde{f}_1(z,\gamma)d\gamma \quad \text{,etc.}$$

With these definitions, Eq. (30) reads

$$-\left[2k_L \frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z) \right] = \mu_0 \frac{\mathrm{K}}{\gamma_0}\frac{\partial j_z}{\partial t} e^{i\Psi} = \mu_0 \frac{\mathrm{K}}{\gamma_0} \frac{\partial\left(j_0 + \tilde{j}_1 e^{i\Psi} + \tilde{j}_1^* e^{-i\Psi} \right)}{\partial t} e^{i\Psi}$$

We use $\Psi = (k_u + k_L)z - \omega_L t$ and assume that $\tilde{\mathrm{j}}_1$ is “almost" independent of time. Then:

$$-\left[2k_L \frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z) \right] \approx \mu_0 \frac{\mathrm{K}}{\gamma_0}\left[(-\mathrm{i}\omega_L)\tilde{j}_1 e^{\mathrm{i}\Psi} + (i\omega_L)\tilde{j}_1^* e^{-\mathrm{i}\Psi} \right] e^{i\Psi} = \mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\left(-\tilde{j}_1 e^{2\mathrm{i}\Psi} + \tilde{j}_1^* \right) \approx \mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\tilde{j}_1^*$$

(neglecting the rapidly term $\tilde{j}_1 e^{2\mathrm{i}\Psi}$). Equally, $\boxed{\mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\tilde{j}_1 = 2k_L \frac{\partial}{\partial \mathrm{z}}\tilde{\mathbf{E}}_0(z)}$ (35)

2.3 Self-consistent description of e.m. field and electron distribution

We can now combine the “field equation” (35) and the “kinematic equation” (34) to find a self-consistent description of the evolution of the e.m. field and the electron density distribution:

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = \mathrm{i}\frac{\mu_0 cK}{2\gamma_0}\tilde{j}_1 = \mathrm{i}\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty \tilde{f}_1(z,\gamma)d\gamma =$$
$$\mathrm{i}\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty d\gamma \int_0^z dz' \left[\mathrm{i}\frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z') \right] \frac{\partial f_0(\gamma)}{\partial\gamma} \exp\left[\mathrm{i}(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z) \right] \tag{36}$$

The problem of this equation is, that it contains not only the desired complex transverse field amplitude $\tilde{\mathbf{E}}_0$, but also the longitudinal space charge field $\tilde{\mathbf{E}}_z$. Fortunately, $\tilde{\mathbf{E}}_z$ can be related to $\tilde{\mathbf{E}}_0$ in the following way: For our assumption of the space charge field $\mathbf{E}_z = \tilde{\mathbf{E}}_z(z)e^{i\Psi} + c.c.$, the longitudinal component of the 1st Maxwell equation reads (note $\frac{\partial}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{y}} = 0$ in our 1D treatment):

$\left(\nabla\times H\right)_z = 0 = j_z + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}_z$ or $\frac{\partial}{\partial t}\mathbf{E}_z\left(z\right) = -\mu_0 c^2 j_z$, thus $\tilde{\mathbf{E}}_z\left(z\right) \approx -\frac{\mathrm{i}\mu_0 c^2}{\omega_L}\tilde{j}_1$. With Eq. (35) this is related to the transverse e.m. field:

$\tilde{j}_1 = -\mathrm{i}\frac{2\gamma_0}{\mu_0 cK}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0\left(z\right)$, thus $\tilde{\mathbf{E}}_z(z) \approx -\frac{2\gamma_0 c}{\omega_L K}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0\left(z\right)$. Therefore Eq. (36) becomes:

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = \mathrm{i}\frac{\mu_0 q^2 K^2 c^2}{4\gamma_0^2 m_0 c^2}\int_1^\infty d\gamma\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma_0^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0\left(z'\right)\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\mathbf{exp}\left[\mathrm{i}(\mathrm{C} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right]$$

This is an integro-differential equation for the complex amplitude of the e.m. field. Only for few non-trivial model functions of the initial energy distribution f_0, the solution can be found analytically, using Laplace transform techniques. We restrict ourselves to the most simple case, a monoenergetic ("cold") beam: $f_0(\gamma) = n_0\delta(\gamma-\gamma_0)$, i.e. $\Delta\gamma=0$, with charge density qn_0, i.e.

$$j_0 = qc\int_{-\infty}^{\infty} n_0\delta(\gamma-\gamma_0)d\gamma = qcn_0 .$$

Integration over energy can then be executed, using partial integration:

$$\int_1^\infty \frac{\mathrm{d}\delta(\gamma-\gamma_0)}{d\gamma}F(\gamma)d\gamma = \left[\delta(\gamma-\gamma_0)F(\gamma)\right]_1^\infty - \int_1^\infty \delta(\gamma-\gamma_0)\frac{dF(\gamma)}{d\gamma}d\gamma ,$$ thus

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = \mathrm{i}\frac{\mu_0 n_0 q^2 K^2}{4\gamma_0^2 m_0}\times$$

$$\int_0^z dz'\int_1^\infty d\gamma\delta(\gamma-\gamma_0)\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right]\left(\mathrm{i}\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}(z'-z)\right)\mathbf{exp}\left[\mathrm{i}(\mathrm{C} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right] =$$

$$= -\frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c}\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma_0^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[\mathrm{iC}(z'-z)\right] =$$

$$-\Gamma^3\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[\mathrm{iC}(z'-z)\right] \qquad (37)$$

with abbreviations:

$$\boxed{\begin{array}{ll} \Gamma^3 = \frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c} = \frac{\pi j_0 K^2(1+K^2)\omega_L}{I_A c\gamma_0^5} & \Gamma \text{ is called gain parameter.} \\ I_A = \frac{4\pi m_0 c}{\mu_0 q} = 17kA \quad \text{is the "Alven current"} & \\ k_p^2 = \frac{4\pi j_0(1+K^2)}{I_A\gamma_0^3} = \Gamma^3\frac{4\gamma^2 c}{\omega_L K^2} & k_p \text{ is the wave number of longitudinal plasma oscillation} \end{array}} .$$

Note that k_p is the only reminder of taking longitudinal space charge into account.

We have ended with an ordinary integro-differential equation (37) for $\tilde{\mathbf{E}}_0$. We now derive Eq. (37)

with respect to z: $\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 = -iC\frac{d}{dz}\tilde{\mathbf{E}}_0 + \Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[iC(z'-z)\right]$, where

$\frac{d}{dz}\int_0^z dz' g(z)h(z') = \frac{d}{dz}\left[g(z)\int_0^z dz'h(z')\right] = \frac{d}{dz}g(z)\int_0^z dz'h(z') + g(z)h(z)$) was used.

Finally, we derive once more and get:

$$\begin{aligned}
\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 &= -iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - iC\Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[iC(z'-z)\right] \\
&= -iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - iC\left(iC\frac{d}{dz}\tilde{\mathbf{E}}_0 + \frac{d^2}{dz^2}\tilde{\mathbf{E}}_0\right) \\
&= -2iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] + C^2\frac{d}{dz}\tilde{\mathbf{E}}_0
\end{aligned}$$

Rearranging, we arrive at our final result: An ordinary linear third-order differential equation for the complex field amplitude $\tilde{\mathbf{E}}_0$:

$$\boxed{\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 + 2iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \left(k_p^2 - C^2\right)\frac{d}{dz}\tilde{\mathbf{E}}_0 = i\Gamma^3\tilde{\mathbf{E}}_0(z)} \qquad (38)$$

At the end of this derivation, Fig. 8 illustrates the major steps and approximations taking us to the final result Eq. (38).

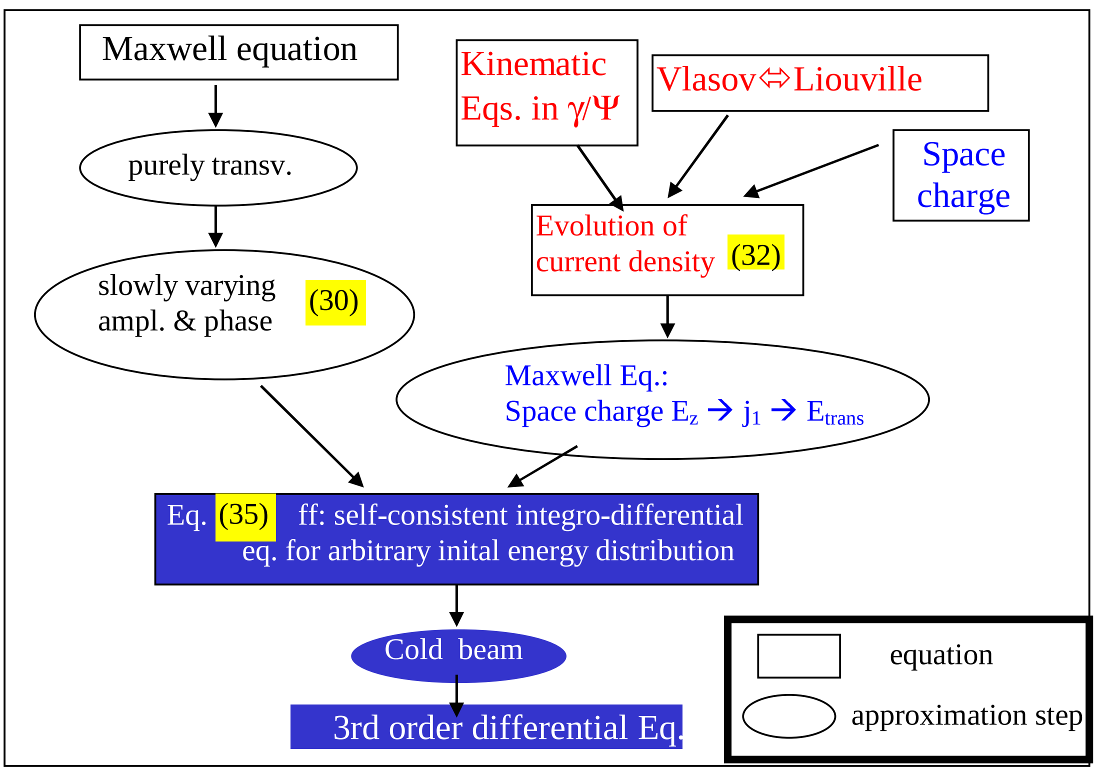

Fig. 8: Major steps to derive the 3rd order differential Eq. (38) for the high-gain free-electron laser

2.4 Solution of the high-gain FEL-equation

For the solution of Eq. (38) we make the Ansatz $\tilde{\mathbf{E}}_0 = A\,\mathbf{exp}(\Lambda z)$ and get the "characteristic equation":

$$\Lambda^3 + 2\mathrm{i}C\Lambda^2 + \left(k_p^2 - C^2\right)\Lambda = \Lambda\left(\Lambda^2 + 2\mathrm{i}C\Lambda - C^2 + k_p^2\right) = \Lambda\left[\left(\Lambda + \mathrm{i}C\right)^2 + k_p^2\right] = \mathrm{i}\Gamma^3 \tag{39}$$

Eq. (39) has three roots, and the the general solution of Eq. (38) is constructed from three independent partial solutions:

$$\tilde{\mathbf{E}}_0(z) = A_1\,\mathbf{exp}(\Lambda_1 z) + A_2\,\mathbf{exp}(\Lambda_2 z) + A_3\,\mathbf{exp}(\Lambda_3 z)\ . \tag{40}$$

The amplitudes A_1, A_2, A_3 are determined by the initial conditions. Since there are three free parameters, we need three independent conditions. The most practical may to specify the these conditions is to specify $\tilde{\mathbf{E}}_0(z=0), \frac{d}{dz}\tilde{\mathbf{E}}_0(z=0), \frac{d^2}{dz^2}\tilde{\mathbf{E}}_0(z=0)$, or, taking into account Eq. (35): $\frac{d}{dz}\tilde{\mathbf{E}}_0 \propto \tilde{j}_1$, to specify $\tilde{\mathbf{E}}_0(z=0), \tilde{j}_1(z=0), \frac{d}{dz}\tilde{j}_1(z=0)$.

We write Eq. (40) in the form $\tilde{\mathbf{E}}_0(z) = A_1\tilde{\mathbf{E}}_1(z) + A_2\tilde{\mathbf{E}}_2(z) + A_3\tilde{\mathbf{E}}_3(z)$, with $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, etc., and we write $\frac{d}{dz}\tilde{\mathbf{E}} = \tilde{\mathbf{E}}'$, etc. (note we will omit the index 0 to $\tilde{\mathbf{E}}_0$ in the following). The general solution, including its first and second derivatives, can then be written in a matrix form: $\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{\mathrm{z}} = \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_{\mathrm{z}} \cdot \begin{pmatrix}\mathrm{A}_1\\ \mathrm{A}_2\\ \mathrm{A}_3\end{pmatrix}$, where the index z means that the matrix elements are taken at longitudinal position *z*. Since $\Lambda_1, \Lambda_2, \Lambda_3$ are known from the characteristic equation (39), all matrix elements are known. Writing the initial condition in the form $\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{\mathrm{z=0}}$, we can calculate A_1, A_2, A_3

from $$\begin{pmatrix}A_1\\ A_2\\ A_3\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_{\mathrm{z=0}}^{-1} \cdot \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0} .$$

Thus, $$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{\mathrm{z}} = \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_{\mathrm{z}} \cdot \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_{\mathrm{z=0}}^{-1} \cdot \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{\mathrm{z=0}}$$

or, using $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, $\tilde{\mathbf{E}}_1'(z) = \Lambda_1\,\mathbf{exp}(\Lambda_1 z)$, etc.,

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} 1 & 1 & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} . \tag{41}$$

Using the explicit expression for the inverse matrix, Eq. (41) reads

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{\Lambda_2\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & -\frac{\Lambda_2+\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & \frac{1}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} \\ \frac{\Lambda_1\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & -\frac{\Lambda_1+\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & \frac{1}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} \\ \frac{\Lambda_2\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & -\frac{\Lambda_2+\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & \frac{1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} \tag{42}$$

2.5 Solution for the case C = k_p = 0

To be more specific, we now investigate the most simple case:

No detuning, i.e. all the electrons have the same energy, and this energy meets exactly the resonance condition: C = 0. Also, we assume negligible impact of space charge, i.e. $k_p^2 = \frac{4\pi j_0 \left(1+K^2\right)}{I_A \gamma_0^3} \to 0$. The validity of this latter condition is a little more difficult to verify and should be considered with care in every specific case. It can be seen that, in tendency, this condition is valid at very high beam energy γ_0. With these assumptions, the three roots of Eq. (39) are:

$$\Lambda^3 = \mathrm{i}\Gamma^3 \Rightarrow \Lambda_1 = -\mathrm{i}\Gamma\mathbf{;}\;\; \Lambda_2 = \frac{\mathrm{i}+\sqrt{3}}{2}\Gamma\mathbf{;}\;\; \Lambda_3 = \frac{\mathrm{i}-\sqrt{3}}{2}\Gamma . \tag{43}$$

The general solution is thus:

$$\begin{aligned} \tilde{\mathbf{E}}(z) &= A_1 \tilde{\mathbf{E}}_1(z) + A_2 \tilde{\mathbf{E}}_2(z) + A_3 \tilde{\mathbf{E}}_3(z) = A_1 \mathbf{exp}(\Lambda_1 z) + A_2 \mathbf{exp}(\Lambda_2 z) + A_3 \mathbf{exp}(\Lambda_3 z) \\ &= A_1 \mathbf{exp}(-\mathrm{i}\Gamma z) + A_2 \mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right) + A_3 \mathbf{exp}\left(\frac{\mathrm{i}-\sqrt{3}}{2}\Gamma z\right) \end{aligned}$$

Obviously, all contributions to this solution either vanish with increasing z, or they oscillate, except for the one containing $A_2 \mathbf{exp}\left(\frac{\sqrt{3}}{2}\Gamma z\right)$. For an undulator much longer than 1/Γ, this part of the solution will dominate.

Using $\Lambda_1, \Lambda_2, \Lambda_3$ from Eq. (43), Eq. (42) reads now (note $1+\mathrm{i}\sqrt{3} = 2\,\mathbf{exp}\,\mathrm{i}\frac{\pi}{3}$):

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3} & \frac{\mathrm{i}}{3\Gamma} & -\frac{1}{3\Gamma^2} \\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-\mathrm{i}\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-\mathrm{i}\frac{\pi}{3}\right) \\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(\mathrm{i}\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(\mathrm{i}\frac{\pi}{3}\right) \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} , \qquad (44)$$

which we will evaluate in the following for two different initial conditions.

2.5.1 Seeding by external electro-magnetic wave at the undulator entrance

First, we consider the case of an external ("seeding") electro-magnetic wave (with amplitude $\mathbf{E}_{ext}$) existing at the undulator entrance, but no initial longitudinal modulation of the electron beam, i.e. $\tilde{j}_1(z=0)=0$. Consequently,

$$\tilde{\mathbf{E}}(z=0)=E_{ext}, \tilde{j}_1(z=0)=0, \ \frac{d}{dz}\tilde{j}_1(z=0)=0 \ \rightarrow \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} = \begin{pmatrix} \mathbf{E}_{ext} \\ 0 \\ 0 \end{pmatrix}.$$ Thus:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2} \\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right) \\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_{ext} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3}\mathbf{E}_{ext} \\ \frac{1}{3}\mathbf{E}_{ext} \\ \frac{1}{3}\mathbf{E}_{ext} \end{pmatrix}$$

$$= \frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix} \tilde{\mathbf{E}}_1+\tilde{\mathbf{E}}_2+\tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1'+\tilde{\mathbf{E}}_2'+\tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1''+\tilde{\mathbf{E}}_2''+\tilde{\mathbf{E}}_3'' \end{pmatrix} = \frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix} \mathbf{exp}(\Lambda_1 z)+\mathbf{exp}(\Lambda_2 z)+\mathbf{exp}(\Lambda_3 z) \\ \Lambda_1\,\mathbf{exp}(\Lambda_1 z)+\Lambda_2\,\mathbf{exp}(\Lambda_2 z)+\Lambda_3\,\mathbf{exp}(\Lambda_3 z) \\ \Lambda_1^2\,\mathbf{exp}(\Lambda_1 z)+\Lambda_2^2\,\mathbf{exp}(\Lambda_2 z)+\Lambda_3^2\,\mathbf{exp}(\Lambda_3 z) \end{pmatrix}$$

Explicitly, the solution for $\tilde{\mathbf{E}}(z)$ is

$$\tilde{\mathbf{E}}(z) = \frac{1}{3}\mathbf{E}_{ext}\left[\mathbf{exp}(-\mathrm{i}\Gamma z)+\mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right)+\mathbf{exp}\left(\frac{\mathrm{i}-\sqrt{3}}{2}\Gamma z\right)\right].$$ As mentioned before, for $z \gg 1/\Gamma$, the solution with Λ_2 dominates:

$$\boxed{\tilde{\mathbf{E}}(z) = \frac{1}{3}\mathbf{E}_{ext}\,\mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right)} \qquad (45)$$

The power gain, defined by $G=\frac{\left|\tilde{\mathbf{E}}\right|^2}{\mathbf{E}_{ext}^2}$, is calculated from Eq. (45) and results in:

$$G=\frac{\left|\tilde{\mathbf{E}}\right|^2}{\mathbf{E}_{ext}^2}=\frac{1}{9}\left[1+4\,\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z\left(\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z+\mathbf{cos}\frac{3}{2}\Gamma z\right)\right]. \qquad (46)$$

For $z \gg 1/\Gamma$, this simplifies to $\boxed{G = \frac{1}{9}\exp\sqrt{3}\Gamma z}$. (47)

The factor 1/9 describes the efficiency at which the incoming ("seeding") electro-magnetic field couples to the FEL gain process. Fig. 8 shows a plot of Eqs. (46, 47) as a function of Γz, indicating that, indeed, the gain grows exponentially according to Eq. (47) for $z \gg 1/\Gamma$. The e-folding length of radiation power is called (power) gain length L_G:

$L_G = \frac{1}{\sqrt{3}\Gamma} = \frac{1}{\sqrt{3}}\left(\frac{I_A c\gamma^5}{\pi j_0 K^2 (1+K^2)\omega_L}\right)^{1/3}$. Using $\omega_L = \frac{4\pi c\gamma^2}{\lambda_u (1+K^2)}$ and expressing the current density $j_0 \approx \frac{\hat{I}}{\pi\sigma_r^2}$ in terms of peak current $\hat{I}$ and beam cross section $\pi\sigma_r^2$, this can be written

$$\boxed{L_G = \frac{1}{\sqrt{3}}\left(\frac{I_A \gamma^3 \sigma_r^2 \lambda_u}{4\pi \hat{I} K^2}\right)^{1/3}} \tag{48}$$

Note that some authors use, instead, the e-folding length for the field amplitude which is $2L_G$.

Another parameter widely used is the dimensionless "FEL-parameter" ρ:

$$\rho = \frac{\lambda_u \Gamma}{4\pi} = \frac{1}{4\pi\sqrt{3}}\frac{\lambda_u}{L_G} = \frac{1}{4\pi\sqrt{3}}\frac{1}{N_{Gain}}. \tag{49}$$

N_{Gain} is the number of undulator periods within one power gain length.

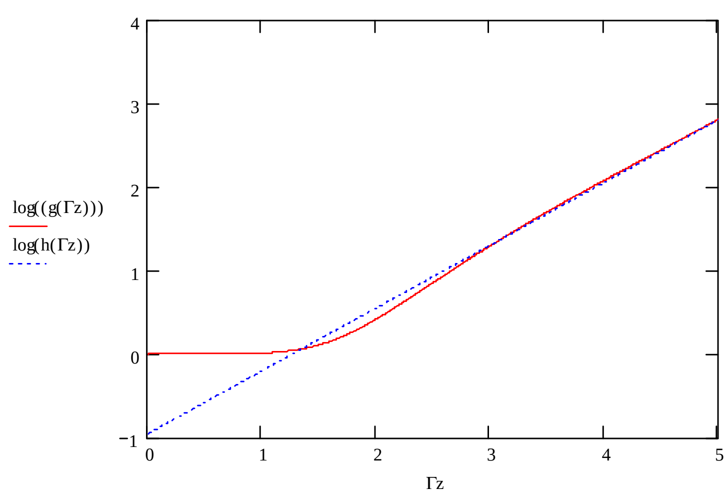

Fig. 8: Plot of the power gain of a high-gain FEL, starting with a seeding e.m. wave, see Eq. (46). The dotted like is the asymptotic solution Eq. (47) for $z \gg 1/\Gamma$. The vertical scale is logarithmic.

2.5.2 Initial longitudinal modulation of electron beam density

As a second example, we consider the case that there is no external e.m. wave at the undulator entrance but a longitudinal current modulation of the electron beam at the radiation wavelength, which is assumed to be stationary at the beginning:

$\tilde{\mathbf{E}}(z=0)=0$, $\tilde{j}_1(z=0)\neq 0$, $\frac{d}{dz}\tilde{j}_1(z=0)=0$. Thus:

$$\tilde{\mathbf{E}}'(z=0)=i\mu_0\frac{cK}{2\gamma_0}\tilde{j}_1(z=0),\ \tilde{\mathbf{E}}''(z=0)=0 \text{ and } \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0}=\begin{pmatrix}0\\ i\mu_0\frac{cK}{2\gamma_0}\tilde{j}_1\\ 0\end{pmatrix}_{z=0}=\begin{pmatrix}0\\ \tilde{\mathbf{E}}'_0\\ 0\end{pmatrix} \tag{50}$$

Therefore:

$$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_z=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3\\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3\end{pmatrix}_z\cdot\begin{pmatrix}\frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2}\\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right)\\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right)\end{pmatrix}\cdot\begin{pmatrix}0\\ \tilde{\mathbf{E}}'_0\\ 0\end{pmatrix}_{z=0}=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3\\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3\end{pmatrix}_z\bullet\begin{pmatrix}\frac{i}{3\Gamma}\tilde{\mathbf{E}}'_0\\ \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0\\ -\frac{1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0\end{pmatrix}$$

Explicitly, the solution for $\tilde{\mathbf{E}}(z)$ is

$$\tilde{\mathbf{E}}(z)=\frac{1}{3\Gamma}\tilde{\mathbf{E}}'_0\left[i\,\mathbf{exp}(\Lambda_1 z)+\mathbf{exp}\left(-i\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_2 z)-\mathbf{exp}\left(i\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_3 z)\right]. \tag{51}$$

Again, for $z\gg 1/\Gamma$, the solution with Λ_2 dominates: $\tilde{\mathbf{E}}(z\gg\frac{1}{\Gamma})\propto\mathbf{exp}\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)$, i.e. we get an exponential growth with the same e-folding length as in the seeding case. The important result is that we don't need any input seeding e.m. wave, a current modulation at the optical, resonant wavelength is as good for starting the process, no matter how small this current modulation is!

From Eq. (50), the radiation power as a function of *z* is calculated:

$$P(z)\propto\left|\tilde{\mathbf{E}}(z)\right|^2\propto\mathbf{cos}\frac{3}{2}\Gamma z\cdot\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z-\sqrt{3}\,\mathbf{sin}\frac{3}{2}\Gamma z\cdot\mathbf{sinh}\frac{\sqrt{3}}{2}\Gamma z+\mathbf{cosh}\sqrt{3}\Gamma z\,. \tag{52}$$

The asymptotic behavior for $z\gg 1/\Gamma$ is $P(z)\propto\mathbf{exp}\sqrt{3}\Gamma z$, very much like in the seeding case. Fig. 9 illustrates both Eq. (52) and its asymptotic behavior.

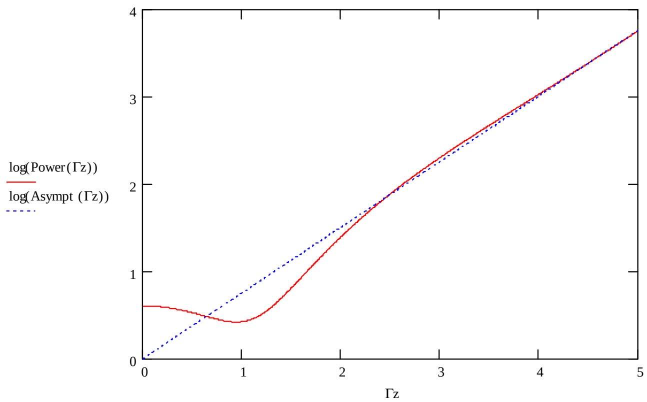

Fig. 9: Plot of the power gain of a high-gain FEL, starting with a longitudinal current modulation of the electron beam at the radiation wavelength, see Eq. (50). The dotted like is the asymptotic solution for $z \gg 1/\Gamma$. The vertical scale is logarithmic.

It is interesting to note that, like in the seeding case, the exponential growth of the e.m. field starts only after approx. three gain lengths, a distance often called "lethargy regime".

2.6 Resonance width

In the previous section we have assumed that all the electrons have the same energy, and this energy meets exactly the resonance condition: $C = 0$. Analysis of the characteristic equation (39) for $k_p = 0$ and $C \neq 0$ is a quite straight-forward algebra. It is seen that

1. The maximum gain occurs indeed for ON-resonance operation (i.e. for $C = 0$). It is important to point out that this behavior is fundamentally in contrast to the low gain case, where no gain was found for particles initially on resonance energy, see Fig. 3.

2. The gain drops significantly when $|C|$ is increased to values corresponding to $\frac{\Delta\gamma}{\gamma} = \rho$.

Because of $\lambda_L \propto \frac{1}{\gamma^2}$, this means the bandwidth of a high-gain FEL is $\boxed{\frac{\Delta\lambda_L}{\lambda_L} = 2\frac{\Delta\gamma}{\gamma} = 2\rho}$ (53)

All particles outside this energy window don't contribute to the gain process constructively. Therefore, the relative energy spread with the electron bunch should be smaller than ρ: $\frac{\Delta\gamma}{\gamma} \leq \rho$. This requirement is a serious technical challenge for FELs operating at low ρ-values. In tendency, this is the case for very short wavelength λ_L. For instance, for the LCLS X-ray FEL presently under construction at SLAC, ρ is approx. 10^{-4}. An comparison of the theoretically expected bandwidth with measurements taken at the short-wavelength FEL at DESY is illustrated in Fig. 10.

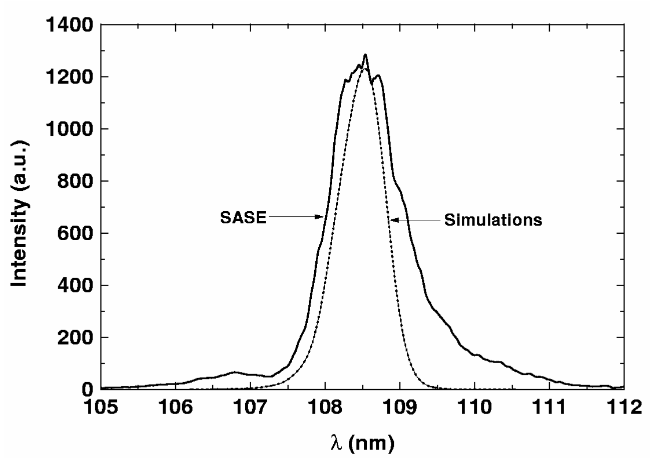

Fig. 10: Wavelength spectrum of the central radiation cone measured at the high-gain FEL at DESY [7], called TTF FEL. The dotted line is the theoretically expected line width.

A different formulation of the same facts is as follows:

A high-gain FEL acts as a narrow-band amplifier with bandwidth $\frac{\Delta\omega}{\omega} \le 2\rho$.

2.7 Laser Saturation

The exponential growth of radiation power will not proceed forever. It comes to an end latest when the electron beam current is perfectly modulated at the optical wavelength. The precise behavior of the high-gain FEL in this saturation regime cannot be treated within our analysis because our linear approximation is based on the assumption $\frac{|\tilde{j}_1|}{j_0} \ll 1$. Some typical features of the saturation regime are as follows: The electrons lose so much energy that they fall out of the resonance condition. Due the bunching and motion in phase space, the e.m. field may even pump back some energy to the electron beam. A potential cure against this is undulator tapering, i.e. increasing the K-parameter to compensate for the loss of electron energy. Also, the energy spread of the electron beam increases (thus the frequency spread of radiation). In any case, the analysis of the non-linear saturation behavior needs numerical simulation and is beyond of the scope of this paper.

However, we are able to perform a simple estimate of the radiation power at saturation: Let's assume $|\tilde{j}_1| = j_0$, i.e. full modulation. With Eq. (35) we estimate the field amplitude at saturation by the assumption that the major part of radiation is generated within the last gain length:

$$\left|\tilde{\mathbf{E}}_0\left(z = L_G\right)\right| \approx \frac{d}{dz}\tilde{\mathbf{E}}_0\left(L_G\right) \times L_G \approx \mu_0 \frac{cK}{2\gamma_0} j_0 L_G$$ Plugging L_G in from Eq. (48) yields

$$\boxed{P_{sat} = \frac{\varepsilon_0 c}{2} |\mathbf{E}|^2 \times Area \approx \frac{\varepsilon_0 c \sigma_r^2}{2} |\mathbf{E}_{sat}|^2 \approx \frac{\mu_0 c}{120} \left(\frac{\hat{I}^2 I_A K \lambda_u}{\sigma_r} \right)^{2/3}}. \tag{54}$$

It is interesting to note that, within this approximation, the saturation power doesn't depend on neither beam energy γ nor radiation wavelength λ_L, very much in contrast to the power of spontaneous undulator radiation. Typical numbers may be:

$\hat{I} = 1000$ A, K=1, $\lambda_u = 0.03\ m$, $\sigma_r = 0.1\ mm \rightarrow P_{sat} \approx 2$ GW

Fig. 12 illustrates onset of FEL saturation at a power level of 1 GW observed at the TTF FEL at DESY/Hamburg with parameters close to these values.

The amount of electron beam power converted to FEL output radiation is called power efficiency and is given by: $\boxed{\frac{P_{sat}}{P_{beam}} = \frac{P_{sat}}{\gamma_o m_0 c^2 \hat{I}/q} \approx \rho}$, i.e. it is just given by the FEL parameter ρ.

As a rule of thumb, saturation sets on after 20 power gain lengths. For the most challenging high gain FEL projects aiming at sub-nanometer wavelengths (e.g. LCLS/SLAC, and the European XFEL/DESY), L_{sat} will be as long as 100-200 m.

2.8 Start-up from noise: Self-Amplified Spontaneous Emission (SASE)

It was found in section 2.5.2 that an arbitrarily small current modulation of the electron beam current at the entrance of the undulator will be sufficient to start the exponential FEL process. Of course, this modulation must be at the resonant radiation wavelength λ_L, determined by the electron energy and undulator parameters λ_u, K, see Eq. (12). For very short wavelengths, (say micrometers or nanometers), this is very difficult to achieve. In fact, because of the narrow-bandpass property described in the previous section, it would be sufficient if the longitudinal electron bunch profile would contain Fourier components at λ_L. However, for normal electron bunch lengths of some 1 mm and λ_L well below a micrometer, this is (practically) not the case.

A very elegant way out is making use of the fact that the electron beam is actually made up of many point-like charges (i.e. electrons) randomly distributed in space and time[5]. Such a random distribution generates a white noise spectrum of current modulation, which always contains some spectral contribution within the FEL bandwidth. This principle was proposed first by Kondratenko and Saldin in 1980 [8] and is widely called the "Self-Amplified Spontaneous Emission" mode (SASE) of high-gain FELs. It is most attractive for very short wavelengths, where no mirrors are available to construct an optical cavity, and no external lasers are available to produce a sufficiently powerful input wave. Tuning of FEL output wavelength is extremely simple in the SASE mode: You just change the electron energy (or, if you prefer, the undulator K-parameter) accordingly, and the SASE process "automatically" selects the correct modulation wavelength from shot noise.

[5] There is a simple proof that this random distribution really exists: It is the basis for the spontaneous undulator radiation. As long as the observed characteristics of spontaneous undulator radiation agree with theoretical expectations, we can safely assume that electrons are distributed randomly.

A characteristic property of SASE FELs is that the output radiation spectrum is noisy, because the FEL amplifies a part of the shot noise spectrum. Fig. 11 illustrates an extreme case of such a noisy output spectrum. The frequency width $\Delta\omega$ of the individual spikes in the output spectrum is determined by the length of the electron bunch l_{bunch} according to $\Delta\omega = \frac{2\pi c}{l_{bunch}}$, i.e. the Fourier transform limit given by the bunch length.

Another important quantity is the number $N_G = \frac{L_G}{\lambda_u}$ of undulator periods within one gain length. Since the radiation pulse slips by one wavelength per undulator period with respect to the electron bunch, it is this quantity N_G which determines the number of wavelengths where coherence is expected within the FEL process. The quantity $l_{coh} \approx N_G \cdot \lambda_L$ is called coherence length. Using Eq. (49), it can be written $l_{coh} \approx \frac{\lambda_L}{\pi\rho}$ (note the factor π comes from a more detailed analysis, Ref. [2]). We would expect that the quantity $\frac{\lambda_L}{l_{coh}} \approx \pi\rho$ should determine the relative bandwidth of the FEL, which is indeed the case, see Eq. (53). If this quantity is larger than $\Delta\omega$, it determines the envelope spectrum containing M spikes in statistical average. In terms of l_{coh}, it is the number of the coherence lengths l_{coh} within the bunch length that determines the average number $M \approx \frac{l_{bunch}}{l_{coh}}$ of spikes within the FEL output spectrum. M is called the number of longitudinal modes.

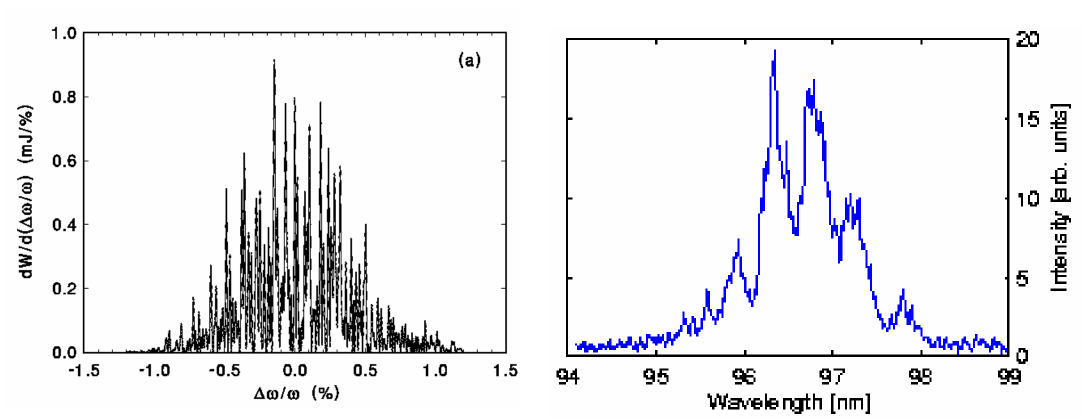

Fig. 11: Since SASE FELs start from shot noise, also the output radiation spectrum is expected to be noisy. In the extreme case of the numerical simulation shown on the left hand side, there is a very large number of spikes (large number of "longitudinal modes") which will fluctuate from electron bunch to electron bunch in intensity within the bandwidth of the FEL. The plot on the right hand side shows measurement at TTF FEL of a single shot spectrum with mode number $M \approx 6$. The envelope of this spectrum corresponds to about $\frac{\Delta\lambda_{ph}}{\lambda_{ph}} \approx 0.015$, in agreement with Fig. 10.

How can we calculate the initial conditions the FEL process is subjected to by the shot noise? One way is to estimate the effective current modulation within the bandwidth $\frac{\Delta\lambda_{ph}}{\lambda_{ph}} = 2\rho$ and use this value as "initial longitudinal current modulation" in the analysis described in section 2.5.2. Another way is to calculate the equivalent input power generated within the first gain length by shot noise and use this value as an external "seed wave" in section 2.5.1. This "effective input power" P_{sh} of shot noise can be estimated at $P_{sh} \approx \frac{3}{N_c\sqrt{\pi \ln N_c}}\rho P_{beam}$ (see Ref. [2], Eq. (6.95)). (55)

Here, P_{beam} is the electron beam power and N_c is 0.5 times the number of electrons within the coherence length. The power gain of a SASE FEL at saturation can be estimated from Eqs. (54,55) at: $G_{sat} = \frac{P_{sat}}{P_{sh}} = \frac{\rho P_{beam}}{P_{sh}} \approx \frac{1}{3} N_c \sqrt{\pi \ln N_c}$, i. e. it is roughly given by the number of electrons in the cooperation length. The quantity P_{sh} is relevant in two ways:

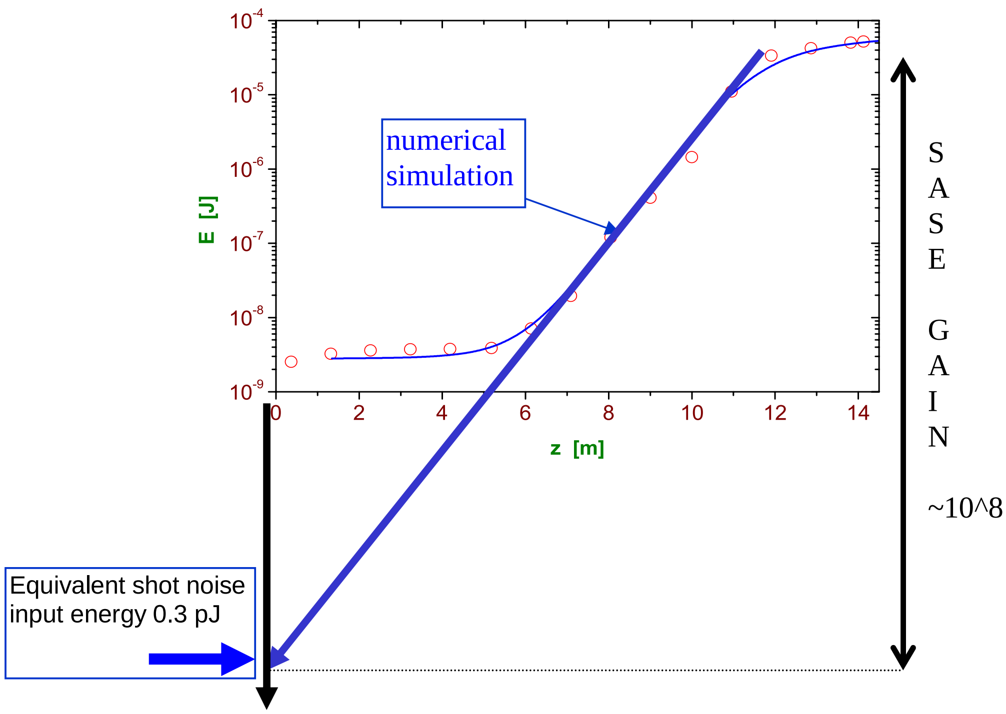

Fig. 12: Energy in the radiation pulse as a function of longitudinal position in the undulator measured at the SASE FEL at DESY at λ_L = 98 nm (dots). The vertical scale is logarithmic. The solid line is the theoretical expectation. If the exponential gain curve is (exponentially) extrapolated down (blue arrow) to the beginning of the undulator, it hits the vertical axis at a value very much in agreement with Eq. (55).

1. Having an estimate for P_{sh} available makes it possible to compare the theoretical SASE model with measurements. Fig. 12 shows the exponential gain observed at the SASE FEL at DESY. Within the first five gain lengths, the measured radiation power is dominated by the spontaneous undulator radiation, so that the start-up process and the lethargy regime cannot directly be observed. However, if the exponential gain curve is (exponentially) extrapolated

down to the beginning of the undulator it hits the vertical axis at a value very much in agreement with Eq. (55). Since P_{sh} is the power amplified by the high-gain FEL, Fig. 12 indicates a total power gain by 8 orders of magnitude, which is indeed about the number of electrons in the cooperation length. Note that this does not mean that the FEL output power exceeds the power of spontaneous undulator radiation by this factor. In contrast, the power of spontaneous undulator radiation may even be comparable to FEL saturation power, but is radiated into a much wider spectrum and opening angle.

2. If one plans to improve the spectral purity of the FEL by using a seeding laser, Eq. (55) provides a lower limit of its required power. If the seed laser power would not exceed P_{sh}, the output radiation would still be determined by shot noise rather by the seed laser spectrum.

2.9 3D effects

Analysis of effects due to the finite transverse size of both the radiation and the electron beam goes beyond the scope of this article. However, some 3D effects have a tremendous practical relevance and will be summarized here in a semi-quantitative way.

2.9.1 Transverse overlap between electron beam and e.m. radiation

The most prominent 3D issue is that the FEL gain process requires complete transverse overlap between the electron beam and the radiation beam during the complete passage of the undulator to ensure that the interaction between e.m. wave and the electron beam takes place as described. Taking into account that, for short-wavelength FEL, the transverse rms beam size is 100 μm or less (see below), this means that the electron orbit must not depart from a perfectly straight line by more than some 10 μm over several gain lengths. This puts stringent tolerances on undulator field errors and is technically difficult both to realize and to verify.

2.9.2 Diffraction

Due to diffraction, even a perfectly coherent plane wave grows in transverse size after a while if it is collimated to a transverse radius of σ_r. The distance $L_R = \frac{\pi\sigma_r^2}{\lambda_L}$ after which the radiation beam is grown by approx. a factor of 2 is called Rayleigh-length and provides an estimate of the opening angle $\sigma_\measuredangle$ of the radiation: $\sigma_\measuredangle \approx \frac{2\sigma_r}{L_R} \approx \frac{\lambda_L}{2\sigma_r}$.

An equivalent estimate comes from the transverse phase space volume covered by a perfectly coherent source known to be $\sigma_r\sigma_\measuredangle = \frac{\lambda_L}{2}$, thus $\sigma_\measuredangle = \frac{\lambda_L}{2\sigma_r}$.

Typical numbers for the LCLS project are $\lambda_L \approx 10^{-10}\,\mathrm{m}$, $\sigma_\mathrm{r} \approx 30\ \mu\mathrm{m}$, yielding $\sigma_\measuredangle \approx 2\ \mu\mathrm{rad}$. It is interesting to note that this value is much smaller than the characteristic opening angle of undulator radiation $\frac{1}{\gamma} \approx 30\ \mu\mathrm{rad}$! The reason is that FEL radiation is no single-charge radiation but is a product of coherent superposition of radiation coming from many electrons distributed in longitudinal direction, very much like an array of antennas is able to generate a directional characteristic of radio wave emission.

Within our FEL analysis we have implicitly assumed that the e.m. wave is transversely coherent during the entire process. This is certainly not the case for SASE. The SASE FEL starts with many transverse optical modes. Since the axial mode achieves the highest gain, it reaches saturation first, so that at saturation "normally" the radiation is almost fully coherent.

2.9.3 Emittance of the electron beam

The emittance of the electron beam introduces a longitudinal velocity spread in the electron beam very much like energy spread does. Thus, in terms of FEL gain, electron emittance is equivalent to addition energy spread. The equivalent energy spread is

$$\left.\frac{\Delta\gamma}{\gamma}\right|_{eff} \approx \frac{\gamma^2\varepsilon}{\beta\left(1+K^2\right)}$$ (β is the Twiss parameter of electron focusing).

With the condition $\frac{\Delta\gamma}{\gamma} < \rho$ derived from Eq. (53) this gives a limit for the beam emittance:

$$\varepsilon \approx \frac{\beta\left(1+K^2\right)}{\gamma^2}\left.\frac{\Delta\gamma}{\gamma}\right|_{\grave{e}ff} < \frac{\beta\left(1+K^2\right)}{2\gamma^2}\rho\,, \tag{56}$$

where the factor 2 makes sure the emittance contributes less than 50% of the effective energy spread budget (if the latter is defined such that it contains both contributions by emittance <u>and</u> momentum spread).

A second condition comes from the diffraction effect: We want to maintain both complete overlap of electron beam and radiation (calling for long L_R thus large σ_r) AND maximum possible gain (calling for small σ_r, thus small L_R) . The best compromise is $L_R \approx L_g$ [6], thus $L_R = \frac{\pi\sigma_r^2}{\lambda_L} = \frac{\pi\beta\varepsilon}{\lambda_L} \approx L_G = \frac{1}{4\pi\sqrt{3}}\frac{\lambda_u}{\rho}$. With the help of Eq. (56), ρ can be eliminated, yielding

$$\varepsilon < \frac{\lambda_L}{2(3)^{1/4}\pi} \approx \frac{\lambda_L}{4\pi}. \qquad \boxed{\varepsilon < \frac{\lambda_L}{4\pi}}$$ is a rather challenging condition for λ_L in the nanometer range.

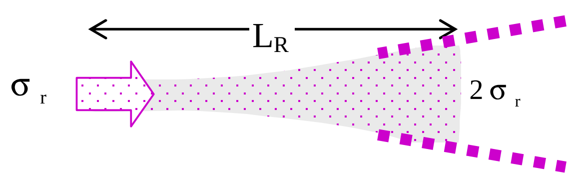

Fig. 13: Sketch of the growth of the transverse size of the radiation beam due to diffraction within a distance called Rayleigh length L_R .

[6] Note that this condition also enables development of transverse coherence in case the process starts from noise (i.e. from a transversely incoherent source like in the SASE case), because it provides transverse mixing of radiation fields originating from different portions of the electron beam.

2.10 Velocities

When we introduced the complex field amplitude $\tilde{\mathbf{E}}_0(z) = \mathbf{E}_0(z)\,\mathbf{exp}\,i\psi_E$ in Sect. 2.1, we have intentionally introduced an additional phase ψ_E. This slowly varying phase describes the slippage of the e.m. phase with respect to a free wave propagating at phase velocity *c*. We can determine $\cos\psi_E$ by (see. Fig. 14):

$$\mathbf{cos}\,\psi_E = \frac{\Re\tilde{\mathbf{E}}}{\left|\tilde{\mathbf{E}}\right|} = \frac{\mathbf{cos}(\Gamma z) + 2\,\mathbf{cos}\left(\frac{\Gamma z}{2}\right)\cdot\mathbf{cosh}\left(\frac{\sqrt{3}}{2}\Gamma z\right)}{\sqrt{1 + 4\,\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z\left(\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z + \mathbf{cos}\frac{3}{2}\Gamma z\right)}}.$$ For $z \gg 1/\Gamma$ this reads

$$\mathbf{cos}\,\psi_E = \frac{\Re\tilde{\mathbf{E}}}{\left|\tilde{\mathbf{E}}\right|} = \frac{\Re\,\mathbf{exp}\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)}{\left|\mathbf{exp}\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)\right|} = \Re\,\mathbf{exp}\left(\frac{i}{2}\Gamma z\right) = \mathbf{cos}\frac{\Gamma z}{2},\ \text{i.e.}\ \psi_E = \frac{\Gamma z}{2} \quad \text{for } z \gg 1/\Gamma.$$

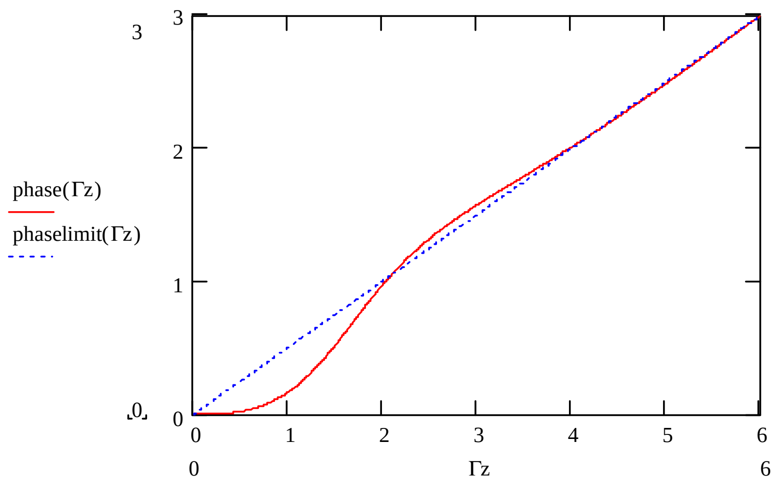

Fig. 14: Development of the slowly varying phase ψ_E as a function of the longitudinal coordinate *z*, normalized to the gain parameter Γ. ψ_E describes the slippage of the e.m. phase with respect to a free wave propagating at phase velocity *c*.

We can now calculate the phase velocity of the e.m. wave during the FEL process:

$v_{ph} = \frac{\omega}{k} = \frac{\omega}{k_L + \frac{\Gamma}{2}}$, i.e. it is reduced by $c - v_{ph} \approx c - \frac{\omega}{k_L}\left(1 - \frac{\Gamma}{2k_L}\right) = \frac{\Gamma}{2k_L}c$ with respect to a free e.m. wave.

In a similar way, we can calculate the phase velocity of density modulation. Due to Eq. (35), and using Eq. (44):

$$j_z = j_0 + \tilde{j}_1 e^{i\Psi} + c.c. = j_0 - i\frac{2\gamma_0}{\mu_0 c K}\frac{1}{3}\mathbf{E}_{ext}\Lambda_2 \,\mathbf{exp}(\Lambda_2 z) e^{i\Psi} + c.c. =$$

$$= j_0 + const.\,\mathbf{exp}\frac{\sqrt{3}\Gamma z}{2}\cdot\mathbf{exp}\,i\left[\left(\frac{\Gamma}{2} + k_u + k_L\right)z - \omega t\right] + c.c.$$

Thus, the phase velocity of the density modulation is given by

$\mathrm{v}_{j1} = \dfrac{\omega}{k_{eff}} = \dfrac{\omega}{k_u + k_L + \frac{\Gamma}{2}} \approx \dfrac{\omega}{k_u + k_L} - \dfrac{\Gamma}{2k_L}c$. Since $\dfrac{\omega}{k_u + k_L}$ is the mean longitudinal velocity of the resonant electrons, it is seen that the growing density modulation slowly slips backwards with respect to the bunch center.

Finally, the group velocity of e.m. wave packets during the FEL process is of interest:

Analyzing how Λ_2 depends on C shows that $\mathrm{v_g} = \dfrac{d\omega}{dk} \approx c\left(1 - \dfrac{1+K^2}{3\gamma_0^2}\right)$.

In conclusion, we can distinguish 4 characteristic velocity slippages with respect to c in the high-gain FEL:

$\mathrm{c} - \mathrm{v_{ph}} = \mathrm{c}\dfrac{\Gamma}{2\mathrm{k_L}}$ with $\mathrm{v_{ph}}$ the phase velocity of e.m. wave during gain process.

$\mathrm{c} - \mathrm{v_g} = \mathrm{c}\dfrac{1+\mathrm{K}^2}{3\gamma_0^2}$ with $\mathrm{v_g}$ the group velocity of e.m. wave during gain process.

$\mathrm{c} - \mathrm{v_z} = \mathrm{c}\dfrac{1+\mathrm{K}^2}{2\gamma_0^2}$ with $\mathrm{v_z}$ the longitudinal velocity of the electron bunch
(i.e. of resonant particles, "kinematical slippage").

$\mathrm{c} - \mathrm{v_{j1}} = \mathrm{c}\left(\dfrac{1+\mathrm{K}^2}{2\gamma_0^2} + \dfrac{\Gamma}{2\mathrm{k_L}}\right)$ with $\mathrm{v_{j1}}$ the phase velocity of density modulation during gain process.

From these relations, we can calculate the slippage $\mathrm{v_g} - \mathrm{v_z}$ of radiation wave packages (“spikes” in time domain) with respect to the electron bunch. It is 3 times smaller than the kinematic slippage: $\dfrac{\mathrm{v_g} - \mathrm{v_z}}{\mathrm{c} - \mathrm{v_z}} = \dfrac{1}{3}$

3. APPENDIX

3.1 How to recover the low gain result from high gain solution

Since the low gain FEL is just a special case of the high gain FEL, it should be possible to recover the properties of low gain FELs, in particular the Madey Theorem (see Section 1.3.3), from high gain solutions. To do so, we present a semi-analytically treatment, closely following J.B. Murphy and C. Pellegrini [9]. It comes in three steps:

1. For the high gain FEL, gain is defined by $G_{\text{high gain}} = \frac{\left|\tilde{\mathbf{E}}(z = L_u)\right|^2}{\left(\mathbf{E}(z = 0)\right)^2}$, so $G_{\text{high gain}} = 1$ if there is no gain at all. In contrast, the gain defined for low gain FELs is $G_{\text{low gain}} = \frac{\left|\tilde{\mathbf{E}}(z = L_u)\right|^2 - \left(\mathbf{E}(z = 0)\right)^2}{\left(\mathbf{E}(z = 0)\right)^2}$, see Eq. (18). Thus, if we want to recover the low gain result, we must investigate $$G_{\text{low gain}} = G_{\text{high gain}} - 1 \tag{57}$$

2. In order to recover Madey's Theorem, we must consider a finite deviation of the electron energy from resonance condition, i.e. the case of non-zero detuning. For simplicity, we assume $k_p = 0$. Thus we must investigate the characteristic equation $\Lambda\left(\Lambda + \mathrm{i}C\right)^2 = \mathrm{i}\Gamma^3$, see Eq. (39). Since we are seeking for a solution for significant detuning $\Delta\gamma/\gamma$ but a very small gain parameter Γ, it is reasonable to define dimensionless, reduced parameters $\hat{C} = \frac{C}{\Gamma}$ and $\hat{\Lambda} = \frac{\Lambda}{\Gamma}$ thus rewriting the characteristic equation

$$\hat{\Lambda}\left(\hat{\Lambda} + \mathrm{i}\hat{C}\right)^2 = \mathrm{i} \ . \tag{58}$$

We will look for a solution valid for $\hat{C} \gg 1$. If we interpret $\Delta\gamma$ as the energy deviation from resonance energy γ_0 and observe the definition of C along with Eq. (13), we see that $C = \frac{2k_u}{\gamma_0}\Delta\gamma$ and $\hat{C} = \frac{C}{\Gamma} = \frac{4\pi\sqrt{3}L_G}{\lambda_u}\frac{\Delta\gamma}{\gamma_0} = \frac{1}{\rho}\frac{\Delta\gamma}{\gamma_0}$. Thus, our assumption $\hat{C} \gg 1$ corresponds to $\rho \ll \Delta\gamma/\gamma$. Approximate solutions to Eq. (58) are given by

$$\hat{\Lambda}_1 \approx \frac{-\mathrm{i}}{\hat{\mathrm{C}}^2} \ ; \ \hat{\Lambda}_2 \approx \mathrm{i}\left(\frac{1}{\sqrt{\hat{\mathrm{C}}}} - \hat{\mathrm{C}}\right) \ ; \ \hat{\Lambda}_3 \approx \mathrm{i}\left(-\frac{1}{\sqrt{\hat{\mathrm{C}}}} - \hat{\mathrm{C}}\right) \tag{59}$$

We now demonstrate that these solutions are indeed good for $\hat{C} \gg 1$ by calculating $F\left(\hat{C}\right) = \hat{\Lambda}_i^3 + 2i\hat{C}\hat{\Lambda}_i^2 - \hat{C}^2\hat{\Lambda}_i - i$ (i = 1,2,3). This quantity should be zero for all $\hat{\Lambda}_i$ satisfying Eq. (58). Indeed, as seen from Fig. 15, $F\left(\hat{C}\right)$ approaches zero for all $\hat{\Lambda}_i$, as long as $\hat{C} = \frac{C}{\Gamma} > 15$.

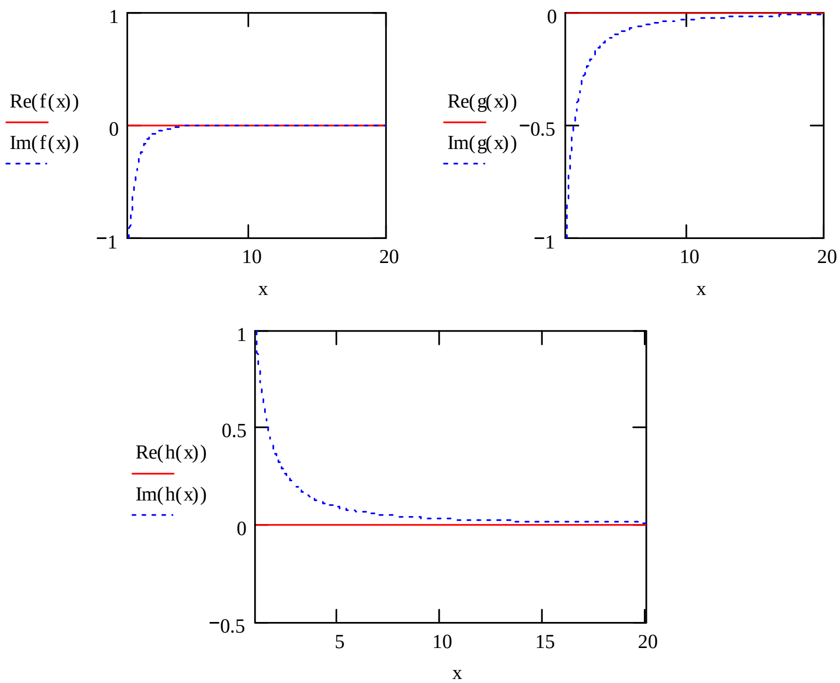

Fig. 15: Plots of $F(\hat{C})$ as a function of $\hat{C}$, for $\Lambda_i = \Lambda_1, \Lambda_2, \Lambda_3$. Note that, in this plot, $x \equiv \hat{C}$ and *f(x), g(x), h(x)* stand for *F(x)* with $\Lambda_i = \Lambda_1, \Lambda_2, \Lambda_3$, respectively. While the real parts are zero anyhow, the imaginary parts approach zero for $x = \hat{C} = \dfrac{C}{\Gamma} > 15$, thus indicating that $\Lambda_i = \Lambda_1, \Lambda_2, \Lambda_3$ as given by Eq. (59) are the three independent solutions to Eq. (58) in this regime.

3. We now use $\Lambda_i = \Lambda_1, \Lambda_2, \Lambda_3$ as given by Eq. (59) to construct the solution $\mathbf{E}(z)$ according to section 2.4. In terms of initial conditions, the low gain FEL corresponds to the case "Seeding by external electro-magnetic wave at the undulator entrance" treated in section 2.5.1, i.,e.

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} = \begin{pmatrix} \mathbf{E}_{ext} \\ 0 \\ 0 \end{pmatrix}$$

. According to Eq. (42) we get

$$\tilde{\mathbf{E}}(z) = \mathbf{E}_{ext}\left[M_{11}\,\mathbf{exp}(\Lambda_1 z) + M_{21}\,\mathbf{exp}(\Lambda_2 z) + M_{21}\,\mathbf{exp}(\Lambda_2 z)\right]$$

with $$M_{11} = \frac{\Lambda_2\Lambda_3}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)},$$ etc. .

We can now calculate the FEL gain in low gain approximation according to Eq. (57) (in analogy to Eq. (46)). The numerical result presented in Fig. 16 indicates that this gain indeed resembles the low gain dependence on energy detuning (Madey's theorem) shown in Fig. 3. The analytical calculation is left to the reader.

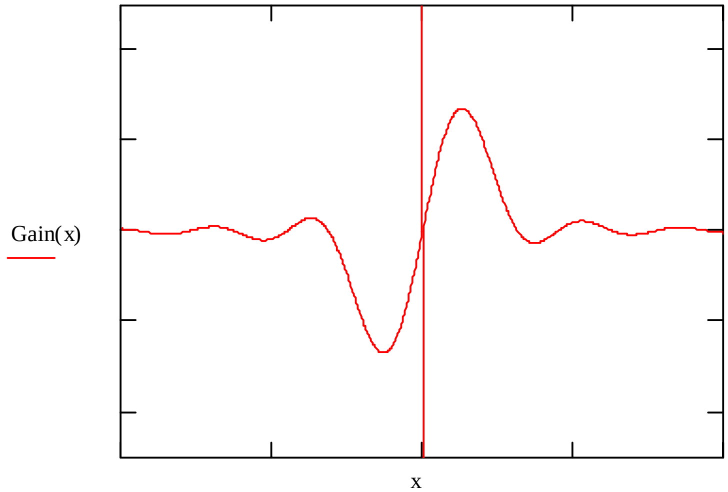

Fig. 16: FEL gain according to Eq. (57) vs. energy detuning, as derived from high gain theory after application of low gain approximation $\hat{C} \equiv x \gg 1$, and using the approximate solutions Eq. (59) valid for $x > 15$. The vertical scale is arbitrary.

ACKNOWLEDGEMENTS

The author acknowledges with pleasure numerous valuable discussions with his colleagues G. Hoffstätter, E.L. Saldin, E.A. Schneidmiller, and M.V. Yurkov. For the chapter on low-gain FELs, unpublished lecture notes by G. Hoffstätter have been used.

REFERENCES

[1] K. Wille, Introduction to accelerator physics, (Teubner Studienbücher, 1992)

[2] E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov, The Physics of Free-Electron Lasers (Springer, 2000).

[3] J.D. Jackson, Classical Electrodynamics (De Gruyter).

[4] P. J. Duke, Synchrotron Radiation (Oxford Science Publications, 2000).

[5] J.M.J. Madey, Relationship between Mean Radiated Energy and Spontaneous Power Spectrum in a Free Electron Laser, Nuovo Cimento B**50**, 64 (1979)

[6] A. Renieri, Nuovo Cim. **53B**, 160 (1979)

[7] J. Andruszkow, et al., First observation of Self-Amlified Spontaneous Emission in a Free-Electron Laser at 109 nm Wavelength, Phys. Rev. Lett. 85, 3825 (2000)

[8] A. M. Kondratenko and E. L. Saldin, Generation of coherent radiation by a relativistic electron beam in an undulator, Part. Accelerators, 10, 207 (1980)

[9] J.B. Murphy, C. Pellegrini, in: Laser Handbook Vol. 6 (North Holland): Introduction to the physics of the FEL