TESLA-FEL 2009-07

Errors in Reconstruction of Difference Orbit Parameters due to Finite BPM-Resolutions

V.Balandin, W.Decking and N.Golubeva

Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

Contents

1 Introduction

The standard approach to determine variation of the transverse beam orbit (transverse jitter) at some location in a (transversely uncoupled) transport line is based on the analysis of beam position measurements. If the optical model of the beam line and BPM resolutions are known, the typical choice is to let difference orbit parameters be a solution of the weighted linear least squares problem. This least squares problem, in the next turn, can be solved analytically resulting in the well known formulas for both, estimates of the difference orbit parameters and covariance matrix of the errors of these estimates (see, for example [1, 2]).

Nevertheless, although analytical solutions are available, their direct usage as a tool for designing of a "good measurement system" does not look to be fairly straightforward. One of the reasons is their dependence from position of the reconstruction point.

In this paper, first, considering propagation of the estimate of the difference orbit parameters and propagation of the covariance matrix of the errors of this estimate along the beam line as the position of the reconstruction point changes, we will introduce error Twiss parameters and invariant error emittance, which will allow us, up to large extend, to separate the role of the optics between BPM locations and the effects coming from the choice of the position of the reconstruction point.

In our approach, the problem of designing new or estimating properties of the existing measurement system can be formulated using the usual accelerator physics concepts of emittance and betatron functions. The magnitude of the reconstruction errors is defined by error emittance and the needed balance between position and momentum reconstruction errors in the point of interest has to be achieved by matching error Twiss parameters in this point to the desired values.

Second, taking into account that for devices as SASE FELs we are mainly interested not in the beam jitter at the device entrance itself, but in the resulting trajectory derivation form some golden orbit along the whole device, we introduce Courant-Snyder invariant as estimator of the reconstruction errors of the difference orbit parameters.

As it could be expected, the reconstruction errors, when measured using Courant-Snyder invariant, do not depend on the position of the reconstruction point, but depend not only on the error emittance but also on the design betatron functions (although our simple BPM readings model do not take information about beam sizes at BPM locations into account). The figure of merit for the quality of the measurement system is now not the error emittance alone, but the product of error emittance and mismatch between error and design Twiss parameters. Large mismatch can spoil the properties of the measurement system even for the case when the error emittance is small.

2 Standard Least Squares Solution

In this section we will review the standard approach to the problem of determination of difference orbit parameters using readings of beam position monitors under assumptions that the optical model of the beam line and BPM resolutions are known. We will assume that the transverse particle motion is uncoupled in linear approximation and will use the variables $\vec{z} = (\,x,\,p\,)^{\top}$ for the description of the horizontal beam oscillations. Here, as usual, x is the horizontal particle coordinate and p is the horizontal canonical momentum scaled with the kinetic momentum of the reference particle. As orbit parameters we will understand values of x and p given in some predefined point in the beam line (reconstruction point with longitudinal position

$s = r$) and would like to obtain estimates of these parameters by fitting BPM data to the known linear model of the beam transport. Because unknown BPM offsets, non-zero corrector strengths and misaligned quadrupoles make the absolute orbit deviate from a simple betatron oscillation, the fit should, in general, work better when fitted not to an absolute orbit, but to a difference orbit. So, again as usual, we will assume that some "golden trajectory" $\bar{x}$ and $\bar{p}$ is given and, instead of the problem of estimating of absolute orbit parameters, will consider the problem of estimating parameters of the difference orbit $\delta\vec{z} = (x - \bar{x},\, p - \bar{p})^{\top}$ in the reconstruction point.

Let us assume that we have n BPMs in our beam line placed at positions $s_1, \ldots, s_n$ and they deliver readings

$$\vec{b}_c \;=\; (b_1^c, \,\ldots\, b_n^c)^{\top} \tag{1}$$

for the current trajectory with previously recorded observations for the golden orbit being

$$\vec{b}_g \;=\; (b_1^g, \,\ldots\, b_n^g)^{\top} \,. \tag{2}$$

Suppose that the difference between these readings can be represented in the form

$$\delta\vec{b}_{\varepsilon} \;\stackrel{\mathrm{def}}{=}\; \vec{b}_c - \vec{b}_g \;=\; \begin{pmatrix} x(s_1) - \bar{x}(s_1) \\ \vdots \\ x(s_n) - \bar{x}(s_n) \end{pmatrix} \;+\; \vec{\varepsilon}\,, \tag{3}$$

where the random vector $\vec{\varepsilon} \;=\; (\varepsilon_1, \,\ldots,\, \varepsilon_n)^{\top}$ has zero mean and positive definite covariance matrix V_{ε}, i.e. that

$$< \vec{\varepsilon} > = \vec{0}, \qquad \mathcal{V}(\vec{\varepsilon}) \;= < \vec{\varepsilon} \cdot \vec{\varepsilon}^{\top} > - < \vec{\varepsilon} > \cdot < \vec{\varepsilon} >^{\top} = V_{\varepsilon}. \tag{4}$$

Note that from (3) and (4) it follows that $\mathcal{V}(\delta\vec{b}_{\varepsilon}) = \mathcal{V}(\vec{\varepsilon})$.

Let $A_m(r)$ be a (symplectic) transfer matrix from location of the reconstruction point to the m-th BPM location

$$A_m(r) \;=\; \begin{pmatrix} a_m(r) & c_m(r) \\ e_m(r) & d_m(r) \end{pmatrix}, \qquad a_m(r)\, d_m(r) \;-\; c_m(r)\, e_m(r) \;\equiv\; 1\,. \tag{5}$$

In order to find estimate $\delta\vec{z}_{\varepsilon}(r) \;=\; (\delta x_{\varepsilon}(r),\, \delta p_{\varepsilon}(r))^{\top}$ for the difference orbit parameters in the presence of BPM reading errors one has to solve a linear system

$$M(r) \cdot \delta\vec{z}_{\varepsilon}(r) \;\stackrel{\mathrm{def}}{=}\; \begin{pmatrix} a_1(r) & c_1(r) \\ \vdots & \vdots \\ a_n(r) & c_n(r) \end{pmatrix} \cdot \delta\vec{z}_{\varepsilon}(r) \;=\; \delta\vec{b}_{\varepsilon}\,. \tag{6}$$

If number of BPMs is grater than two or/and rank of the matrix M is smaller than two, the system (6) may not have classical solutions at all or may have many of them. In this case "solve" means to find a vector $\delta\vec{z}_\varepsilon$ such that $M \cdot \delta\vec{z}_\varepsilon$ is the "best" approximation to the measured vector $\delta\,\vec{b}_\varepsilon$. There are many possible ways of defining the "best" solution. A choice which can often be motivated for statistical reasons (Gauss-Markov theorem and its generalization; see, for example, [3]) and also leads to a simple computational problem is to let $\delta\vec{z}_\varepsilon$ be a solution to the minimization problem

$$\min_{\delta\vec{z}_\varepsilon} \; \left(M \cdot \delta\vec{z}_\varepsilon \, - \, \delta\vec{b}_\varepsilon \right)^\top V_\varepsilon^{-1} \left(M \cdot \delta\vec{z}_\varepsilon \, - \, \delta\vec{b}_\varepsilon \right) . \tag{7}$$

Let us assume that the Cholesky factorization $V_\varepsilon = R_\varepsilon^\top R_\varepsilon$ of the covariance matrix can be computed. Then

$$\left(M \cdot \delta\vec{z}_\varepsilon \, - \, \delta\vec{b}_\varepsilon \right)^\top V_\varepsilon^{-1} \left(M \cdot \delta\vec{z}_\varepsilon \, - \, \delta\vec{b}_\varepsilon \right) \; = \; \left\| R_\varepsilon^{-\top} \left(M \cdot \delta\vec{z}_\varepsilon \, - \, \delta\vec{b}_\varepsilon \right) \right\|_2^2 \tag{8}$$

and the minimization problem (7) becomes the weighted linear least squares problem

$$\min_{\delta\vec{z}_\varepsilon} \; \left\| R_\varepsilon^{-\top} M \cdot \delta\vec{z}_\varepsilon \, - \, R_\varepsilon^{-\top} \cdot \delta\vec{b}_\varepsilon \right\|_2^2 . \tag{9}$$

Here $\| \cdot \|_2$ denotes the Euclidean vector norm and R_ε is a (unique) upper triangular matrix with strictly positive diagonal elements.

The problem (9) always has at least one solution and the solution with minimal Euclidean norm (which is unique) is given by the formula

$$\delta\vec{z}_\varepsilon \; = \; \left(R_\varepsilon^{-\top} M \right)^\dagger R_\varepsilon^{-\top} \cdot \delta\,\vec{b}_\varepsilon \, , \tag{10}$$

where $\left(R_\varepsilon^{-\top} M \right)^\dagger$ is the pseudoinverse (the Moore-Penrose pseudoinverse) of the matrix $R_\varepsilon^{-\top} M$.

If the matrix $R_\varepsilon^{-\top} M$ has full column rank, then the solution of the problem (9) is unique. Matrix R_ε is nondegenerated, that means that the rank of the matrix $R_\varepsilon^{-\top} M$ is always equal to the rank of the matrix M. Because due to symplecticity of the matrices A_m

$$a_m^2 \; + \; c_m^2 \; \neq \; 0 \, , \tag{11}$$

the matrix M always has at least n nonzero elements and therefore its rank never can be equal to zero. So if $\mathrm{rank}(M) < 2$, then it must be equal to one, leading to the following observation:

First, $\mathrm{rank}(M) = 1$ if and only if columns of M are linearly dependent, i.e. if there exist constants ξ and η with $\xi^2 + \eta^2 \neq 0$ such that for all m simultaneously

$$\xi \, a_m \; + \; \eta \, c_m \; = \; 0 \, . \tag{12}$$

Second, let us introduce B_{mk} - transport matrix from the location of the BPM with index m to the location of the BPM with index k

$$B_{mk} \;=\; \begin{pmatrix} æ_{11}^{mk} & æ_{12}^{mk} \\ æ_{21}^{mk} & æ_{22}^{mk} \end{pmatrix} \;=\; A_k\, A_m^{-1} \;=\; \begin{pmatrix} d_m\, a_k - e_m\, c_k & a_m\, c_k - c_m\, a_k \\ d_m\, e_k - e_m\, d_k & a_m\, d_k - c_m\, e_k \end{pmatrix}. \tag{13}$$

From (13) one sees, that if (12) is satisfied, then all $æ_{12}^{mk} \;=\; 0$, meaning that the phase advance between any pair of beam position monitors is multiple of 180°. The reverse statement is also true: if the phase advance between at least two BPMs is not multiple of 180°, then $\mathrm{rank}(M) = 2$.

In the following we will assume that the matrix M has full column rank, which is equivalent to the condition that the matrix $M^{\top} V_{\varepsilon}^{-1} M$ is positive definite and is equivalent to the condition that there are at least two beam position monitors with phase advance between them different from $k \times 180°$. In this case the solution of the problem (9) is unique and the pseudoinverse can be calculated using the regular inverse of the matrix $M^{\top} V_{\varepsilon}^{-1} M$

$$\left(R_{\varepsilon}^{-\top} M\right)^{\dagger} \;=\; \left(M^{\top} V_{\varepsilon}^{-1} M\right)^{-1} M^{\top} R_{\varepsilon}^{-1}. \tag{14}$$

Finally, combining (10) and (14) we obtain the following formula for the estimate of the difference orbit parameters

$$\delta \vec{z}_{\varepsilon}(r) \;=\; \left(M^{\top}(r) V_{\varepsilon}^{-1} M(r)\right)^{-1} M^{\top}(r)\, V_{\varepsilon}^{-1} \cdot \delta\, \vec{b}_{\varepsilon}\,, \tag{15}$$

and the covariance matrix of the errors of this estimate (which can be computed from (15) using (4)) is given by

$$V_z(r) \;\stackrel{\mathrm{def}}{=}\; \mathcal{V}\,(\,\delta \vec{z}_{\varepsilon}(r)\,) \;=\; \left(M^{\top}(r)\, V_{\varepsilon}^{-1}\, M(r)\right)^{-1}. \tag{16}$$

To finish this section let us, for the case when readings of different BPMs are uncorrelated, i.e. when the covariance matrix V_{ε} is a positive diagonal matrix

$$V_{\varepsilon} \;=\; \mathrm{diag}\left(\sigma_1^2,\, \sigma_2^2,\, \ldots,\, \sigma_n^2\right) \;>\; 0\,, \tag{17}$$

rewrite the expression for the error covariance matrix (16) in the more familiar (for example, from [2]) form

$$V_z(r) \;=\; \frac{1}{\Delta} \begin{pmatrix} \sum\limits_{m=1}^{n} \left(\frac{c_m(r)}{\sigma_m}\right)^2 & -\sum\limits_{m=1}^{n} \left(\frac{a_m(r)}{\sigma_m}\right)\left(\frac{c_m(r)}{\sigma_m}\right) \\ -\sum\limits_{m=1}^{n} \left(\frac{a_m(r)}{\sigma_m}\right)\left(\frac{c_m(r)}{\sigma_m}\right) & \sum\limits_{m=1}^{n} \left(\frac{a_m(r)}{\sigma_m}\right)^2 \end{pmatrix}, \tag{18}$$

where

$$\Delta \;=\; \frac{1}{2} \sum_{k,m=1}^{n} \left(\frac{a_k(r) c_m(r) - a_m(r) c_k(r)}{\sigma_k \sigma_m}\right)^2 \;=\; \frac{1}{2} \sum_{k,m=1}^{n} \left(\frac{æ_{12}^{km}}{\sigma_k \sigma_m}\right)^2 \tag{19}$$

is independent from the position of the reconstruction point.

3 Error Emittance and Error Twiss Parameters

The formulas (15) and (16) (and also (18) and (19)) are analytical solutions to the problem considered, but their direct usage as a tool for designing of a "good measurement system" does not look to be fairly straightforward. One of the reasons is their dependence from position of the reconstruction point. In this section, considering propagation of the estimate of the difference orbit parameters and propagation of the covariance matrix of the errors of this estimate along the beam line as the position of the reconstruction point changes, we will introduce error Twiss parameters and invariant error emittance, which will allow us, up to large extend, to separate the role of the optics between BPM locations and the effects coming from the choice of the position of the reconstruction point.

Let $A(r_1, r_2)$ be a matrix which transport particle coordinates from the point with the longitudinal position $s = r_1$ to the point with the longitudinal position $s = r_2$. Using that for all BPM location

$$A_m(r_2) \;=\; A_m(r_1)\, A^{-1}(r_1,\, r_2) \tag{20}$$

we have

$$M(r_2) \;=\; M(r_1)\, A^{-1}(r_1,\, r_2)\,. \tag{21}$$

This equality, when combined with formula (15), gives us

$$\delta\vec{z}_{\varepsilon}(r_2) \;=\; A(r_1,\, r_2)\cdot\delta\vec{z}_{\varepsilon}(r_1)\,, \tag{22}$$

i.e. the estimate of the difference orbit parameters propagates along the beam line exactly as particle trajectory as one changes the position of the reconstruction point.

This fact is more or less obvious and does not lead us to deeper conclusions. The situation becomes less trivial, when from (21) and (16) we obtain

$$V_z(r_2) \;=\; A(r_1,\, r_2)\, V_z(r_1)\, A^{\top}(r_1,\, r_2) \tag{23}$$

and see that the error covariance matrix propagates like the matrix of the second order moments of the beam distribution. It immediately allow us to introduce an error emittance

$$\epsilon_{\varepsilon} = \sqrt{\det V_z(r)} \tag{24}$$

and define error Twiss parameters

$$\begin{pmatrix} \beta_{\varepsilon}(r) & -\alpha_{\varepsilon}(r) \\ -\alpha_{\varepsilon}(r) & \gamma_{\varepsilon}(r) \end{pmatrix} \stackrel{\text{def}}{=} \frac{1}{\epsilon_{\varepsilon}}\, V_z(r)\,. \tag{25}$$

So we see that the properties of new or existing measurement system can be formulated using the usual accelerator physics concepts of emittance and betatron functions. The magnitude of the reconstruction errors is defined by the error emittance, which is an invariant with respect to the choice of the reconstruction point and depend only on transfer matrices between BPM locations and BPM resolutions, and the needed balance between position and momentum errors in the point of interest has to be achieved by matching error Twiss parameters in this point to the desired values.

Let us return to the practically important situation when BPM readings can be considered as uncorrelated. In this case the error emittance is given by the following expression

$$\epsilon_\varepsilon \;=\; \frac{1}{\sqrt{\frac{1}{2}\sum\limits_{k,m=1}^{n}\left(\frac{æ_{12}^{km}}{\sigma_k\sigma_m}\right)^2}}\,, \tag{26}$$

and the error Twiss parameters are

$$\beta_\varepsilon(r) \;=\; \epsilon_\varepsilon \sum_{m=1}^{n}\left(\frac{c_m(r)}{\sigma_m}\right)^2, \qquad \gamma_\varepsilon(r) \;=\; \epsilon_\varepsilon \sum_{m=1}^{n}\left(\frac{a_m(r)}{\sigma_m}\right)^2, \tag{27}$$

$$\alpha_\varepsilon(r) \;=\; \epsilon_\varepsilon \sum_{m=1}^{n}\left(\frac{a_m(r)}{\sigma_m}\right)\left(\frac{c_m(r)}{\sigma_m}\right). \tag{28}$$

What is interesting about the error Twiss parameters (27) and (28) is the fact that they are not simply one of many betatron functions which could propagate through our beam line, they are by themselves solutions of some minimization problem. Let $\beta(r)$, $\alpha(r)$ and $\gamma(r)$ be some Twiss parameters given in the point $s = r$ and let S be a weighted sum

$$S\left(\beta(r),\, \alpha(r),\, \gamma(r)\right) \;=\; \sum_{m=1}^{n}\frac{\beta(s_m)}{\sigma_m^2}\,. \tag{29}$$

Then, under the assumption that the phase advance between at least two BPMs is not a multiple of 180°, the error Twiss parameters are unique solutions to the minimization problem

$$\min_{\beta(r),\,\alpha(r),\,\gamma(r)} S\left(\beta(r),\, \alpha(r),\, \gamma(r)\right) \tag{30}$$

and

$$S\left(\beta_\varepsilon(r),\, \alpha_\varepsilon(r),\, \gamma_\varepsilon(r)\right) \;=\; \sum_{m=1}^{n}\frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \;=\; \frac{2}{\epsilon_\varepsilon}\,. \tag{31}$$

The proof is straightforward and follows from analysis of the explicit representation

$$S\left(\beta(r),\, \alpha(r),\, \gamma(r)\right) \;=$$

$$= \sum_{m=1}^{n} \left(\frac{a_m(r)}{\sigma_m} \right)^2 \beta(r) - 2 \sum_{m=1}^{n} \left(\frac{a_m(r)}{\sigma_m} \right) \left(\frac{c_m(r)}{\sigma_m} \right) \alpha(r) + \sum_{m=1}^{n} \left(\frac{c_m(r)}{\sigma_m} \right)^2 \gamma(r), \quad (32)$$

which can easily be obtained from the equation

$$\beta(s_m) \; = \; a_m^2(r)\, \beta(r) \; - \; 2a_m(r)\, c_m(r)\, \alpha(r) \; + \; c_m^2(r)\, \gamma(r)\,. \quad (33)$$

Because the minimum in (30) can be achieved only in a single point, the equality (31) by itself is important characteristic of the error Twiss parameters. Rewriting it in the form

$$0 \; = \; \left(\sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \right)^2 \; - \; \frac{4}{\epsilon_\varepsilon^2} \; =$$

$$= \; \left(\sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \right)^2 \; - \; 2 \sum_{k,m=1}^{n} \frac{\beta_\varepsilon(s_k)}{\sigma_k^2} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \sin^2\left(\mu_\varepsilon(s_k, s_m)\right) \; =$$

$$= \; \sum_{k,m=1}^{n} \frac{\beta_\varepsilon(s_k)}{\sigma_k^2} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \cos\left(2\mu_\varepsilon(s_k, s_m)\right) \; =$$

$$= \; \sum_{k,m=1}^{n} \frac{\beta_\varepsilon(s_k)}{\sigma_k^2} \cos\left(2\mu_\varepsilon(r, s_k)\right) \cdot \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \cos\left(2\mu_\varepsilon(r, s_m)\right) \; +$$

$$+ \; \sum_{k,m=1}^{n} \frac{\beta_\varepsilon(s_k)}{\sigma_k^2} \sin\left(2\mu_\varepsilon(r, s_k)\right) \cdot \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \sin\left(2\mu_\varepsilon(r, s_m)\right) \; =$$

$$= \; \left(\sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \cos\left(2\mu_\varepsilon(r, s_m)\right) \right)^2 \; + \; \left(\sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \sin\left(2\mu_\varepsilon(r, s_m)\right) \right)^2, \quad (34)$$

where $\mu_\varepsilon(r, s_m)$ is the phase advance calculated from the point $s = r$ to the point $s = s_m$, we obtain that the error betatron functions (and only they) satisfy

$$\sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \cos\left(2\mu_\varepsilon(r, s_m)\right) \; = \; \sum_{m=1}^{n} \frac{\beta_\varepsilon(s_m)}{\sigma_m^2} \sin\left(2\mu_\varepsilon(r, s_m)\right) \; = \; 0 \quad (35)$$

for an arbitrary choice of the point $s = r$ in the beam line[1].

[1] In fact, it is sufficient to check that the equalities (35) are correct only for one particular value of r, then for all other values they will be satisfied automatically.

It seems that the error emittance and error Twiss parameters could be very useful tools for the interpretation of the analytical solution for the covariance matrix (16) in terms of usual accelerator physics notions. In the end of this section, let us give one more example of such interpretation. Namely, let us give an expression for the nonzero singular values $\lambda_{\pm}$ of the matrix $R_{\varepsilon}^{-\top} M$ entering the least squares minimization problem (9)

$$\lambda_{\pm}(r) = \sqrt{\frac{1}{\epsilon_{\varepsilon}} \left(\frac{\beta_{\varepsilon}(r) + \gamma_{\varepsilon}(r)}{2} \pm \sqrt{\left(\frac{\beta_{\varepsilon}(r) + \gamma_{\varepsilon}(r)}{2} \right)^2 - 1} \right)}, \qquad \lambda_{-}\lambda_{+} = \frac{1}{\epsilon_{\varepsilon}}. \tag{36}$$

From (36) one sees that these singular values are nothing else as reciprocals of the lengths of the half-axes of the one sigma error ellipse

$$\gamma_{\varepsilon}(r)\, x^2 \;+\; 2\alpha_{\varepsilon}(r)\, xp \;+\; \beta_{\varepsilon}(r)\, p^2 \;=\; \epsilon_{\varepsilon}\, . \tag{37}$$

4 Courant-Snyder Invariant as Error Estimator

Taking into account that for such devices as, for example, SASE FELs we are mainly interested not in the beam jitter at the device entrance itself, but in the resulting trajectory derivation form some golden orbit along the whole device, it seems almost natural to estimate errors of the reconstruction of the difference orbit parameters not with the help of the covariance matrix (16), but using a single number - the expected value of the Courant-Snyder invariant after substitution into it as arguments the deviation of the estimate of the difference orbit parameters $\delta \vec{z}_{\varepsilon}$ from true difference orbit parameters $\delta \vec{z}_0$.

Let $\beta_0(r)$, $\alpha_0(r)$ and $\gamma_0(r)$ be the design betatron functions and

$$I_x(\, r,\, x,\, p\,) \;=\; \gamma_0(r)\, x^2 \;+\; 2\alpha_0(r)\, xp \;+\; \beta_0(r)\, p^2 \tag{38}$$

be the corresponding Courant-Snyder invariant. The calculation of the desired mean value becomes straightforward, if one takes into account the definition of the error Twiss functions (25) and the property that $\delta \vec{z}_0 \;=\; \langle \delta \vec{z}_{\varepsilon} \rangle$, namely we have

$$\langle\, I_x(\, r,\; \delta x_{\varepsilon} - \delta x_0\, ,\, \delta p_{\varepsilon} - \delta p_0\,)\, \rangle \;=\; \langle\, I_x(\, r,\; \delta x_{\varepsilon} - \langle \delta x_{\varepsilon} \rangle\; ,\, \delta p_{\varepsilon} - \langle \delta p_{\varepsilon} \rangle\,)\, \rangle \;=$$

$$=\; \epsilon_{\varepsilon}\; (\, \gamma_0(r)\, \beta_{\varepsilon}(r) \;-\; 2\alpha_0(r)\, \alpha_{\varepsilon}(r) \;+\; \beta_0(r)\, \gamma_{\varepsilon}(r)\,) \;=\; 2\epsilon_{\varepsilon}\, m_p(\beta_{\varepsilon},\, \beta_0)\, , \tag{39}$$

where

$$m_p(\beta_{\varepsilon},\, \beta_0) \;=\; \frac{\gamma_0(r)\, \beta_{\varepsilon}(r) \;-\; 2\alpha_0(r)\, \alpha_{\varepsilon}(r) \;+\; \beta_0(r)\, \gamma_{\varepsilon}(r)}{2} \tag{40}$$

is the parameter characterizing mismatch between design and error betatron functions (mismatch parameter).

As it was expected, the reconstruction errors, when estimated using Courant-Snyder invariant, do not depend on the position of the reconstruction point, but depend only on the error emittance, the error Twiss parameters and the design betatron functions. The figure of merit for the quality of the measurement system is now not the error emittance alone, but the product of error emittance and mismatch between error and design Twiss parameters. Large mismatch can spoil the properties of the measurement system even for the case when error emittance is small.

Let us give an explicit formula for the mismatch parameter for the case when BPM readings can be considered as uncorrelated. In order to obtain this formula we will use the fact that the design and error betatron functions at the BPM locations can be connected through the relation

$$\beta_0(s_m) \,=\, \beta_\varepsilon(s_m)\cdot\Big(m_p(\beta_\varepsilon,\,\beta_0) + \sqrt{m_p^2(\beta_\varepsilon,\,\beta_0)-1}\cdot\cos\left(2\mu_\varepsilon\left(r,\,s_m\right)-2\theta\right)\Big)\,, \quad (41)$$

where $\theta \,=\, \theta(r,\,\beta_\varepsilon,\,\beta_0)$ is the mismatch phase. Summing up both sides of (41) over all BPMs with weights σ_m^{-2} we have

$$\sum_{m=1}^{n}\frac{\beta_0(s_m)}{\sigma_m^2} \,=\, m_p(\beta_\varepsilon,\,\beta_0)\cdot\sum_{m=1}^{n}\frac{\beta_\varepsilon(s_m)}{\sigma_m^2}\,+$$

$$+\,\sqrt{m_p^2(\beta_\varepsilon,\,\beta_0)-1}\cdot\left(\sum_{m=1}^{n}\frac{\beta_\varepsilon(s_m)}{\sigma_m^2}\cos\left(2\mu_\varepsilon\left(r,s_m\right)\,-\,2\theta\right)\right)\,. \quad (42)$$

Because due to (35) the second term in the right hand side of (42) is equal to zero, the mismatch parameter can be expressed through design and error betatron functions and BPM resolutions as follows

$$m_p(\beta_\varepsilon,\,\beta_0) \,=\, \frac{\sum\limits_{m=1}^{n}\frac{\beta_0(s_m)}{\sigma_m^2}}{\sum\limits_{m=1}^{n}\frac{\beta_\varepsilon(s_m)}{\sigma_m^2}} \,=\, \frac{\epsilon_\varepsilon}{2}\sum_{m=1}^{n}\frac{\beta_0(s_m)}{\sigma_m^2}\,. \quad (43)$$

So, in the case when BPM readings can be considered as uncorrelated, we have

$$\langle\, I_x(\,r,\;\delta x_\varepsilon - \delta x_0\,,\,\delta p_\varepsilon - \delta p_0\,)\,\rangle \,=\, 2\epsilon_\varepsilon\,\frac{\sum\limits_{m=1}^{n}\frac{\beta_0(s_m)}{\sigma_m^2}}{\sum\limits_{m=1}^{n}\frac{\beta_\varepsilon(s_m)}{\sigma_m^2}} \,=\, \epsilon_\varepsilon^2\sum_{m=1}^{n}\frac{\beta_0(s_m)}{\sigma_m^2}\,. \quad (44)$$

5 Worst Case Errors

In the two previous sections we have shown that the statistical properties of the BPM measurement system (the error covariance matrix (16) and the mean value (39) of the Courant-Snyder invariant (38)) can naturally be expressed in terms of error emittance, error Twiss parameters and mismatch between design and error betatron functions, and in this section we will show that factors entering estimates of deviation of reconstructed difference orbit parameters from true difference orbit parameters through the norm of the error vector of BPM readings can also be written using the same quantities. Note that we will make such estimates not with respect to the norm of the error vector $\vec{\varepsilon}$, but with respect to the norm of the normalized error vector

$$\vec{\eta} \;=\; R_{\varepsilon}^{-\top} \cdot\, \vec{\varepsilon}\,, \tag{45}$$

which also has zero mean, but whose covariance matrix is equal to the identity matrix and, therefore, whose components could be considered as "being better balanced in the order of magnitude".

According to (10) the deviation of the reconstructed difference orbit parameters from true difference orbit parameters has the form

$$\delta\vec{z}_{\varepsilon} - \delta\vec{z}_{0} \;=\; \left(R_{\varepsilon}^{-\top} M\right)^{\dagger} R_{\varepsilon}^{-\top} \cdot \left(\delta\,\vec{b}_{\varepsilon} - \delta\,\vec{b}_{0}\right) \;=\; \left(R_{\varepsilon}^{-\top} M\right)^{\dagger} \cdot\, \vec{\eta}\,, \tag{46}$$

and estimates of the norm of this deviation can be obtained using singular values (36) of the matrix $R_{\varepsilon}^{-\top} M$ as follows

$$\|\, \delta\vec{z}_{\varepsilon}(r) - \delta\vec{z}_{0}(r)\,\|_2 \;\le\; \frac{1}{\lambda_{-}(r)} \cdot \|\, \vec{\eta}\,\|_2 \;=\; \epsilon_{\varepsilon}\lambda_{+}(r) \cdot \|\, \vec{\eta}\,\|_2 \;=$$

$$= \sqrt{\epsilon_{\varepsilon}\left(\frac{\beta_{\varepsilon}(r)+\gamma_{\varepsilon}(r)}{2} + \sqrt{\left(\frac{\beta_{\varepsilon}(r)+\gamma_{\varepsilon}(r)}{2}\right)^{2} - 1}\right)} \cdot \|\, \vec{\eta}\,\|_2\;. \tag{47}$$

For the case when the Courant-Snyder invariant is used as error estimator, we have

$$I_x(\,r,\;\delta x_{\varepsilon} - \delta x_0\,,\,\delta p_{\varepsilon} - \delta p_0\,) \;=\; (\delta\vec{z}_{\varepsilon} - \delta\vec{z}_{0})^{\top}\,\Sigma(r)\,(\delta\vec{z}_{\varepsilon} - \delta\vec{z}_{0}) \;=$$

$$=\; \vec{\eta}^{\top} \left(\left(R_{\varepsilon}^{-\top} M\right)^{\dagger}\right)^{\top} \Sigma(r) \left(\left(R_{\varepsilon}^{-\top} M\right)^{\dagger}\right) \vec{\eta}\,. \tag{48}$$

Representing matrix Σ in the form of the product $T^{\top}T$

$$\Sigma \;=\; \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \;=\; \begin{pmatrix} 1/\sqrt{\beta_0} & 0 \\ \alpha_0/\sqrt{\beta_0} & \sqrt{\beta_0} \end{pmatrix}^{\top} \begin{pmatrix} 1/\sqrt{\beta_0} & 0 \\ \alpha_0/\sqrt{\beta_0} & \sqrt{\beta_0} \end{pmatrix} \stackrel{\text{def}}{=} T^{\top}T \tag{49}$$

and introducing matrix

$$K \;=\; T\left(R_{\varepsilon}^{-\top} M\right)^{\dagger} \;=\; T V_z M^{\top} R_{\varepsilon}^{-1} \tag{50}$$

we can rewrite (48) as

$$I_x(r,\ \delta x_\varepsilon - \delta x_0\,,\, \delta p_\varepsilon - \delta p_0\,) \;=\; \vec{\eta}^\top K^\top(r) K(r)\,\vec{\eta}\,. \tag{51}$$

From (51) it is clear that, in order to obtain the desired estimate, we have to calculate the largest eigenvalue of the matrix $K^\top K$. This matrix is n by n matrix, but because all its nonzero eigenvalues coincide with the nonzero eigenvalues of the 2 by 2 matrix

$$K(r)K^\top(r) \;=\; T(r)V_z(r)T^\top(r)\,, \tag{52}$$

the calculations become almost trivial. Namely, we have

$$\mu_\pm \;=\; \frac{\mathrm{tr}\left(KK^\top\right)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}\,(KK^\top)}{2}\right)^2 - \det\,(KK^\top)}\,, \tag{53}$$

where

$$\mathrm{tr}\left(KK^\top\right) \;=\; 2\epsilon_\varepsilon m_p\,(\beta_\varepsilon,\,\beta_0)\;, \qquad \det\left(KK^\top\right) \;=\; \epsilon_\varepsilon^2\,. \tag{54}$$

Substituting (54) into (53) we obtain the explicit representation for two nonzero eigenvalues of the matrix $K^\top K$ through error emittance and mismatch between design and error betatron functions in the form

$$\mu_\pm \;=\; \epsilon_\varepsilon\left(m_p\,(\beta_\varepsilon,\,\beta_0) \pm \sqrt{m_p^2\,(\beta_\varepsilon,\,\beta_0) - 1}\right)\;, \tag{55}$$

and the desired estimate for deviation measured using Courant-Snyder invariant get the following final form

$$I_x(r,\ \delta x_\varepsilon - \delta x_0\,,\, \delta p_\varepsilon - \delta p_0\,) \le \epsilon_\varepsilon\left(m_p\,(\beta_\varepsilon,\,\beta_0) + \sqrt{m_p^2\,(\beta_\varepsilon,\,\beta_0) - 1}\right)\cdot \|\,\vec{\eta}\|_2^2\,. \tag{56}$$

Note that both, (47) and (56), are exact upper estimates, i.e. for certain error vectors $\vec{\eta}$ they become equalities, and it is the reason for us to name these estimates as estimates of worst case errors.

6 Two BPM Case

Let us consider two BPMs separated in the beam line by a transfer matrix

$$B_{12} \;=\; \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \qquad r_{12} \;\neq\; 0 \tag{57}$$

and assume that these BPMs deliver uncorrelated readings with rms resolutions σ_1 and σ_2 respectively. The error emittance of this simplest measurement system is given by

$$\epsilon_\varepsilon \;=\; \frac{\sigma_1\,\sigma_2}{|r_{12}|}\,, \tag{58}$$

and the error Twiss parameters at BPM locations can be calculated as follows

$$\beta_\varepsilon(s_1) \;=\; \frac{\sigma_1}{\sigma_2}\,|r_{12}|\;, \qquad \alpha_\varepsilon(s_1) \;=\; \frac{\sigma_1}{\sigma_2}\,\mathrm{sign}\,(r_{12})\;r_{11}, \tag{59}$$

$$\beta_\varepsilon(s_2) \;=\; \frac{\sigma_2}{\sigma_1}\,|r_{12}|\;, \qquad \alpha_\varepsilon(s_2) \;=\; -\frac{\sigma_2}{\sigma_1}\,\mathrm{sign}\,(r_{12})\;r_{22}. \tag{60}$$

Representing the r_{12} coefficient of the matrix B_{12} in the form

$$r_{12} \;=\; \sqrt{\beta_\varepsilon(s_1)\,\beta_\varepsilon(s_2)}\;\sin\left(\mu_\varepsilon\,(s_1,\,s_2)\right) \;=\; |r_{12}|\,\sin\left(\mu_\varepsilon\,(s_1,\,s_2)\right)\,, \tag{61}$$

we obtain that the sine of the error phase advance is always equal to plus or minus one

$$\sin\left(\mu_\varepsilon\,(s_1,\,s_2)\right) \;=\; \mathrm{sign}\,(r_{12}) \;=\; \pm 1\,, \tag{62}$$

that means that the error phase advance itself is always equal to an odd multiple of 90°.

If we assume that

$$|\mathrm{tr}\,(B_{12})| \;<\; 2\,, \tag{63}$$

then the matrix B_{12} allows periodic beam transport with the periodic betatron function being

$$\beta_p \;=\; \frac{|r_{12}|}{\sqrt{1-\left(\frac{r_{11}+r_{22}}{2}\right)^2}} \;=\; \frac{r_{12}}{\sin\left(\mu_p\,(s_1,\,s_2)\right)}\,, \tag{64}$$

where $\mu_p\,(s_1,\,s_2)$ is the corresponding phase advance. Calculating according to (43) the mismatch between the error and the periodic Twiss parameters we obtain

$$m_p\,(\beta_\varepsilon,\,\beta_p) \;=\; \frac{1}{2}\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1}\right)\cdot\frac{\sin\left(\mu_\varepsilon\,(s_1,\,s_2)\right)}{\sin\left(\mu_p\,(s_1,\,s_2)\right)}\,. \tag{65}$$

From this formula we see, for example, that if BPMs have different resolutions, the usage of periodic betas as design betas will not lead to optimal measurement conditions when using Courant-Snyder invariant as an error estimator even in the case when periodic beam transport has a phase advance odd multiples of 90°.

7 Error Emittance in Periodic Systems

In this section we will consider a measurement system constructed from n identical cells assuming that the cell transfer matrix allows periodic beam transport with phase advance per cell μ_p corresponding to the periodic betatron function β_p being not a multiple of $180°$. Additionally, we will assume that BPMs placed in our beam line deliver uncorrelated readings, all with the same rms resolution σ_{bpm}.

Let us first consider the case when we have one BPM per cell (identically positioned in all cells) with the periodic betatron function at the BPM locations equal to $\beta_p(s_1)$. In this situation the formula for the error emittance is rather simple and is given by the following expression

$$\epsilon_\varepsilon \;=\; \frac{2\sigma_{bpm}^2}{n\beta_p(s_1)} \cdot m_p(\beta_\varepsilon,\, \beta_p)\,, \tag{66}$$

where

$$m_p(\beta_\varepsilon,\, \beta_p) \;=\; \left(1 - \left(\frac{1}{n} \cdot \frac{\sin(n\mu_p)}{\sin(\mu_p)}\right)^2\right)^{-\frac{1}{2}} \tag{67}$$

is the mismatch between the error and the periodic betatron functions (even so we do not assume, in general, periodic betatron functions being the design betatron functions matched to our beam line).

The expressions for the error Twiss parameters at the BPM locations are given below

$$\beta_\varepsilon(s_k) \;=\; \left(1 - \frac{1}{n} \cdot \frac{\sin(n\mu_p)}{\sin(\mu_p)}\; \cos\left((n+1-2k)\,\mu_p\right)\right) m_p(\beta_\varepsilon,\, \beta_p) \cdot \beta_p(s_1), \tag{68}$$

$$\alpha_\varepsilon(s_k) \;=\; m_p(\beta_\varepsilon,\, \beta_p) \cdot \alpha_p(s_1) \,+$$

$$+\frac{1}{n} \cdot \frac{\sin(n\mu_p)}{\sin(\mu_p)}\, m_p(\beta_\varepsilon,\, \beta_p) \left(\sin\left((n+1-2k)\,\mu_p\right) - \alpha_p(s_1)\cos\left((n+1-2k)\,\mu_p\right)\right), \tag{69}$$

and one sees that while the error beta function always have mirror symmetry

$$\beta_\varepsilon(s_k) \;=\; \beta_\varepsilon(s_{n+1-k})\,, \qquad k \;=\; 1,\, \ldots,\, n, \tag{70}$$

the error alpha function will be mirror antisymmetric

$$\alpha_\varepsilon(s_k) \;=\; -\alpha_\varepsilon(s_{n+1-k})\,, \qquad k \;=\; 1,\, \ldots,\, n \tag{71}$$

only in the case when $\alpha_p(s_1) \;=\; 0$. Note, for completeness, that the mean values of error beta and alpha functions always satisfy the following, quite similar relations

$$\frac{1}{n}\sum_{k=1}^{n} \beta_\varepsilon(s_k) \;=\; \frac{\beta_p(s_1)}{m_p\left(\beta_\varepsilon,\, \beta_p\right)}, \qquad \frac{1}{n}\sum_{k=1}^{n} \alpha_\varepsilon(s_k) \;=\; \frac{\alpha_p(s_1)}{m_p\left(\beta_\varepsilon,\, \beta_p\right)}. \tag{72}$$

Looking at the formulas (66) and (67) one may conclude that the error emittance as a function of the periodic phase advance will be minimized if

$$\sin(n\mu_p) = 0\,, \tag{73}$$

i.e. when the phase advance per cell obey that "common rule stating that the optimal phase advance must be equal 180° divided by n", and when error Twiss parameters coincide with the periodic Twiss parameters. This, of course, is true, if we are free in choosing the cell phase advance while the beta function at the BPM location has to stays unchanged. But it is not, in general, the case when we optimize the phase advance of a cell with a fixed magnetic structure. To be more specific, let us consider a thin lens FODO cell of the length L (where drift sections, not bending magnets, separate the focusing and defocusing lenses) as a basic unit of our periodic system. Let us also assume that the BPM is placed in the "center" of the focusing lens with the beta function at this locations being

$$\beta_p(s_1) \;=\; \beta_+ \;=\; L\,\frac{1+\sin(\mu_p/2)}{\sin(\mu_p)}. \tag{74}$$

In this situation the quality of our measurement system, when estimated using Courant-Snyder invariant with periodic Twiss parameters being the design Twiss parameters, has the form

$$\langle I_x \rangle \;=\; 2\epsilon_\varepsilon m_p\left(\beta_\varepsilon,\, \beta_p\right) \;=\; \frac{4\sigma_{bpm}^2}{nL} \cdot \Psi_n\left(\mu_p\right)\,, \tag{75}$$

where

$$\Psi_n\left(\mu_p\right) \;=\; \frac{\sin(\mu_p)}{1+\sin(\mu_p/2)} \cdot m_p^2\left(\beta_\varepsilon,\, \beta_p\right)\,. \tag{76}$$

All functions $\Psi_n\left(\mu_p\right)$ become unbounded when μ_p approaches points $0°$ and 180°, but inside this interval they converge (from above) to the function

$$\Psi_\infty\left(\mu_p\right) \;=\; \frac{\sin(\mu_p)}{1+\sin(\mu_p/2)} \tag{77}$$

as n goes to infinity. The functions $\Psi_n\left(\mu_p\right)$ for $n = 2, 3, 4, 5$ are plotted in figure 1 together with their values in the points

$$\mu_p \;=\; k\,\frac{180°}{n}\,, \qquad k = 1, \ldots, n-1, \tag{78}$$

shown as small circles at the corresponding curves. One sees that there is nothing really special about points (78) except, of course, trivial fact that all of them belong to the graph of the function Ψ_∞. What could be more interesting, it is the fact that the point of the global minimum moves closer and closer to $180°$ as n increases.

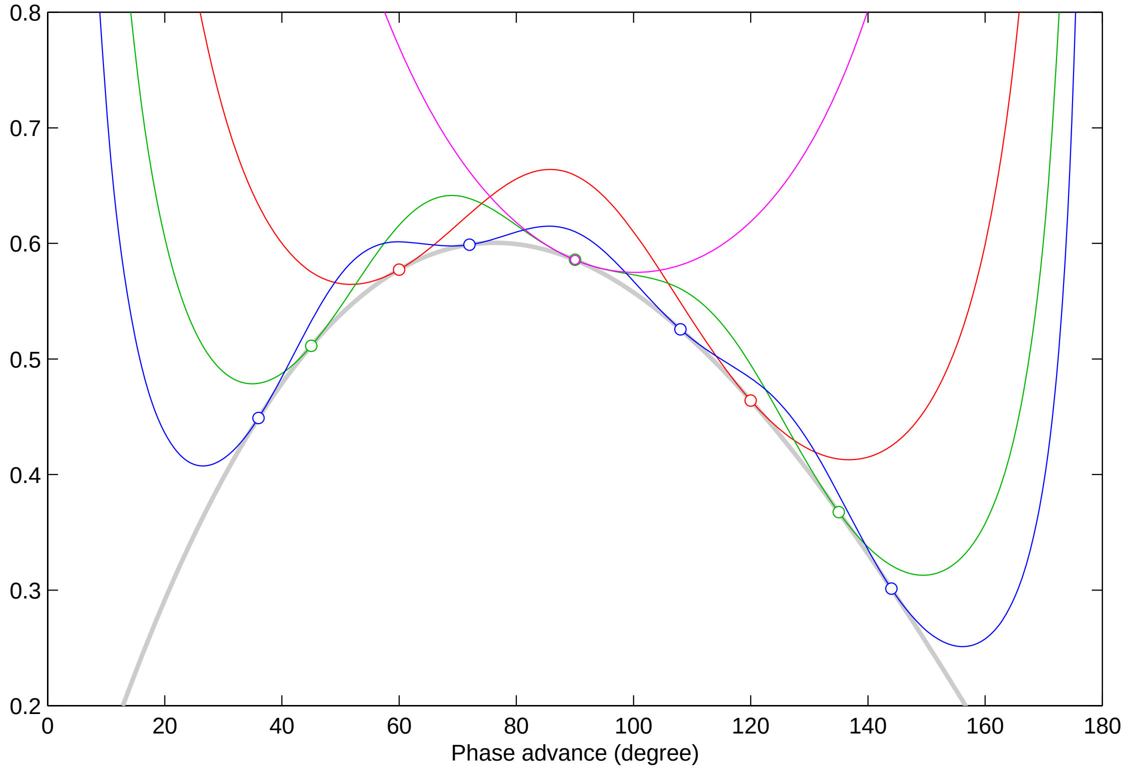

Figure 1: Functions $\Psi_n(\mu_p)$ shown for $n = 2, 3, 4, 5$ (magenta, red, green and blue curves respectively). The gray curve shows function $\Psi_\infty(\mu_p)$.

Let us now turn to the situation when we have two BPMs per cell with θ being the phase shift between the first and second BPM location. In this situation the error emittance can be expressed as

$$\epsilon_\varepsilon \;=\; \frac{2\sigma_{bpm}^2}{n\left(\beta_p(s_1)+\beta_p(s_2)\right)}\cdot m_p(\beta_\varepsilon,\,\beta_p)\,, \tag{79}$$

where

$$m_p(\beta_\varepsilon,\,\beta_p) \;=\; \left(1\;-\;\left(1\;-\;4\sin^2(\theta)\frac{\beta_p(s_1)\beta_p(s_2)}{\left(\beta_p(s_1)+\beta_p(s_2)\right)^2}\right)\cdot\left(\frac{1}{n}\cdot\frac{\sin(n\mu_p)}{\sin(\mu_p)}\right)^2\right)^{-\frac{1}{2}} \tag{80}$$

is the mismatch between error and periodic betatron functions.

For a thin lens FODO cell with BPMs placed in the "centers" of focusing and defocusing lenses we have $\theta \;=\; \mu_p/2$ and the periodic beta function at the BPM locations is equal to

$$\beta_\pm \;=\; L\,\frac{1\pm\sin(\mu_p/2)}{\sin(\mu_p)}\,. \tag{81}$$

With this assumption the formulas for the error emittance and for the mismatch between error and periodic Twiss parameters become

$$\epsilon_\varepsilon \;=\; \frac{\sigma_{bpm}^2}{n\,L}\,\sin(\mu_p)\cdot m_p(\beta_\varepsilon,\,\beta_p) \tag{82}$$

and

$$m_p(\beta_\varepsilon,\,\beta_p) \;=\; \left(1\;-\;\left(1\;-\;\frac{1}{4}\sin^2(\mu_p)\right)\cdot\left(\frac{1}{n}\cdot\frac{\sin(n\mu_p)}{\sin(\mu_p)}\right)^2\right)^{-\frac{1}{2}}\,. \tag{83}$$

Again, we can write

$$\langle I_x\rangle \;=\; 2\epsilon_\varepsilon m_p\left(\beta_\varepsilon,\,\beta_p\right) \;=\; \frac{4\sigma_{bpm}^2}{nL}\cdot\Phi_n\left(\mu_p\right)\,, \tag{84}$$

where

$$\Phi_n\left(\mu_p\right) \;=\; 0.5\sin(\mu_p)\cdot m_p^2\left(\beta_\varepsilon,\,\beta_p\right)\,. \tag{85}$$

As can be seen in figure 2, the situation becomes more symmetric in comparison with figure 1, but still nothing special can be concluded about points (78) (except for the case when $n=2$). Note also that though we are using two times larger number of BPMs, the error resolution does not become two times better.

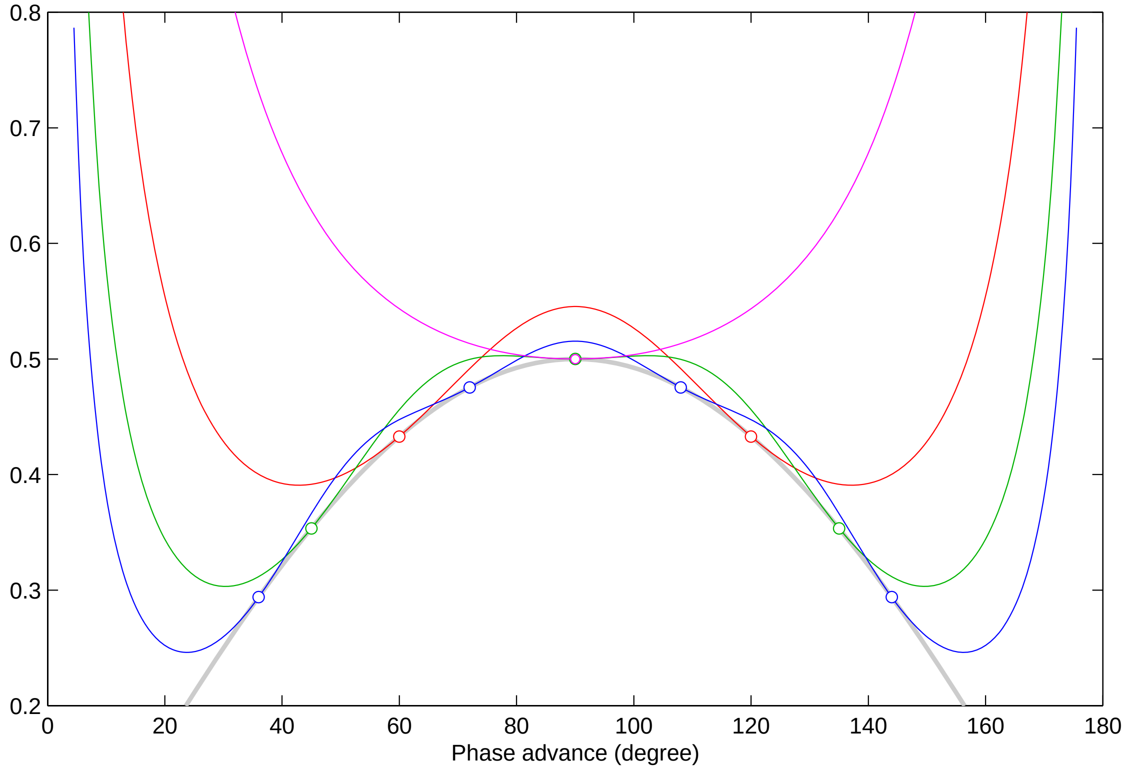

Figure 2: Functions $\Phi_n(\mu_p)$ shown for $n = 2, 3, 4, 5$ (magenta, red, green and blue curves respectively). The gray curve shows function $\Phi_\infty(\mu_p) = 0.5\sin(\mu_p)$.

8 Acknowledgments

This study was motivated by the ongoing design of the high-energy intra bunch train feedback system (IBFB) for the European X-Ray FEL Free-Electron Laser (XFEL) Facility [4, 5]. With very tight tolerances on the trajectory jitter along the SASE undulators, a high precision in the determination of the difference orbit parameters becomes one of the important components of the feedback system, and we thank Boris Keil for bringing our attention to this problem and for stimulating discussions.

References

[1] T.Lohse and P.Emma, "Linear Fitting of BPM Orbits and Lattice Parameters", SLAC-CN-371 (1989).

[2] Y.Chao, "Optics Measurement Resolution and BPM Errors", Proc. PAC 1997, Vancouver, B.C., Canada, p.2125.

[3] A.Björck, Numerical Methods for Least Squares Problems, SIAM, 1996.

[4] M.Altarelli, R.Brinkmann et al. (ed) "XFEL: European X-Ray Free-Electron Laser Facility - Technical Design Report", DESY 2006-097, July 2006.

[5] V.Balandin, W.Decking, N.Golubeva, "Magnet Lattice for High-Energy XFEL IBFB: Version of May 2008", Unpublished Note.